IMPEDANCES AND POWER LOSSES FOR AN OFF-AXIS BEAM

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Abstract

A method for calculating coupling impedances and power losses for off-axis beams is developed. It is applied to calculate impedances of small localized discontinuities like holes and slots, as well as the impedance due to a finite resistivity of chamber walls, in homogeneous chambers with an arbitrary shape of the chamber cross section. The approach requires to solve a two-dimensional electrostatic problem, which can be easily done numerically in the general case, while for some particular cases analytical solutions are obtained.

1 INTRODUCTION

The beam-chamber coupling impedances, as well as power losses due to a finite conductivity of the chamber wall, may depend essentially on the beam position inside the chamber. While for the power loss in a circular pipe this dependence is well-known [1], developing an approach working for other chamber cross sections seems to be worthwhile.

In the present note, we consider the problem for the vacuum chamber with an arbitrary but constant cross section, and calculate, for an off-axis beam, the coupling impedance due to either resistive wall or a small localized discontinuity, like a hole. Analytical results are presented for circular and rectangular cross sections.

2 LONGITUDINAL IMPEDANCE

Let us consider an infinite cylindrical chamber with an arbitrary cross section S. The z axis is directed along the chamber axis, an ultrarelativistic point charge q moves parallel to the axis with the transverse offset $\vec{a}$ from it. A small discontinuity (e.g., a hole) located on the chamber wall at the point $(\vec{b}, z=0)$, contributes as an inductance to the longitudinal coupling impedance [2, 3]

$$Z(k;\vec{a}\,) = -ikZ_0 e_\nu^2(\vec{a}\,)\,(\psi-\chi)\,/2\,, \qquad (1)$$

where $Z_0 = 120\pi$ Ohms is the impedance of free space, $k = \omega/c$, and ψ and χ are magnetic and electric polarizabilities of the discontinuity. The dependence on the beam position, as well as on the hole position in the cross section, is via

$$e_\nu(\vec{a}\,) = -\sum_s k_s^{-2} e_s(\vec{a}\,)\nabla_\nu e_s(\vec{b}\,) \qquad (2)$$

which is merely the normalized electrostatic field produced at the hole location by a filament charge displaced from the chamber axis by distance $\vec{a}$. It is expressed in terms of eigenvalues k_{nm}^2 and orthonormalized eigenfunctions (EFs) $e_{nm}(\vec{r})$ of the 2D boundary problem in S:

$$\left(\nabla^2 + k_{nm}^2\right) e_{nm} = 0\,; \qquad e_{nm}\big|_{\partial S} = 0\,. \qquad (3)$$

We denote $\hat{\nu}$ and $\hat{\tau}$ the outward normal and tangent unit vectors to the boundary ∂S of the chamber cross section S, so that $\{\hat{\nu}, \hat{\tau}, \hat{z}\}$ form a right-handed basis. One should note the normalization condition

$$\oint_{\partial S} dl\; e_\nu = 1\,, \qquad (4)$$

where integration goes along the boundary ∂S, which reflects the Gauss law. It follows from the fact that Eq. (2) gives the boundary value of $\vec{e}_\nu(\vec{a}\,) \equiv -\vec{\nabla}\Phi(\vec{r}-\vec{a}\,)$, where $\Phi(\vec{r}-\vec{a}\,)$ is the Green function of boundary problem (3): $\nabla^2\,\Phi(\vec{r}-\vec{a}\,) = -\delta(\vec{r}-\vec{a}\,)$. For the symmetric case of an on-axis beam in a circular pipe of radius b from Eq. (4) immediately follows $e_\nu(0) = 1/(2\pi b)$.

Likewise, a finite resistivity of the chamber wall leads to the resistive impedance per unit length of the chamber, e.g. [4],

$$Z_L(k;\vec{a}\,)/L = Z_s(k)\oint_{\partial S} dl\; e_\nu^2(\vec{a}\,)\,, \qquad (5)$$

where the surface impedance $Z_s(k)$ is equal to $Z_0 k\delta/2$ when skin-depth δ is smaller than the wall thickness.

Therefore, the problem of the impedance dependence on the beam position is reduced to evaluating $e_\nu(\vec{a}\,)$, cf. Eqs. (1) and (5). It can be performed analytically for simple cross sections when the EFs are known, or numerically in a general case, applying any 2D electrostatic code and imposing (4) for normalization of a numerical solution.

3 BEAM-POSITION DEPENDENCE

3.1 *Circular Chamber*

Using known eigenfunctions (e.g., [5] or see [6]) for a circular cross section of radius b, we sum up in Eq. (2) to get

$$e_\nu(\vec{a}\,) = \frac{1}{2\pi b}\,\frac{b^2-a^2}{b^2-2ab\cos(\varphi_a-\varphi_h)+a^2}\,. \qquad (6)$$

Here a is the beam offset, φ_a, φ_h are azimuth positions of the beam and hole. Result (6) coincides with the known distribution of the wall current, e.g. [7]. Figure 1 shows the

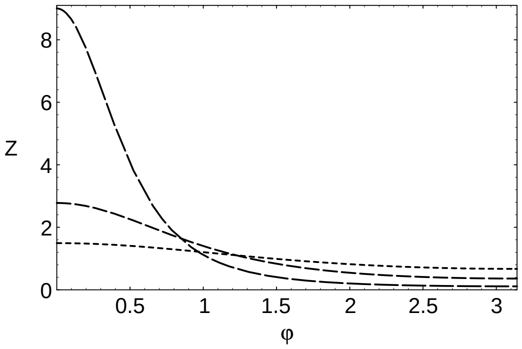

Figure 1: Impedance of a hole in circular pipe versus azimuth angle $\varphi = \varphi_a - \varphi_h$ between beam and hole (in radians) for different beam offsets $a/b = 0.1$ (short-dashed), 0.25, and 0.5 (long-dashed). $Z = 1$ corresponds to $a = 0$ (on-axis beam).

beam-position dependence of the hole impedance (1). Integrating in (5) yields the well-known beam-position dependence for the power loss, e.g. [1],

$$\oint_{\partial S} dl\; e_\nu^2(\vec{a}\,) = \frac{1}{2\pi b}\,\frac{b^2 + a^2}{b^2 - a^2}\,. \tag{7}$$

3.2 Rectangular Chamber

The eigenvalues and EFs for a rectangular chamber of width w and height h are well known, see in [5] or [6]. Let a hole be located in the side wall at $x = w$, $y = y_h$. Then from Eq. (2) for the beam offset $\vec{a} = (x, y)$; ($|x| \le w/2$, $|y| \le h/2$) from the axis at $(w/2, h/2)$ follows

$$\begin{aligned} e_\nu(\vec{a}\,) = \frac{2}{h}\Bigg[&\sum_{n=0}^{\infty} (-1)^n \sin\frac{(2n+1)\pi y_h}{h} \times \\ &\cos\frac{(2n+1)\pi y}{h}\,\frac{\sinh[(n+1/2)\pi(w+2x)/h]}{\sinh[(2n+1)\pi w/h]} \\ &+ \sum_{n=1}^{\infty} (-1)^n \sin\frac{2n\pi y_h}{h}\sin\frac{2n\pi y}{h}\times \\ &\frac{\sinh[n\pi(w+2x)/h]}{\sinh[2n\pi w/h]}\Bigg]\,. \end{aligned} \tag{8}$$

Despite a rather long expression, this series is fast convergent and convenient for evaluations, and it looks much simpler for a centered beam, with $x = y = 0$, cf. [3]. Figure 2 shows that the impedance increases significantly as the beam is displaced closer to the hole.

For integrated e_ν^2 we obtain

$$\begin{aligned} \oint_{\partial S} dl\; e_\nu^2(\vec{a}\,) = \frac{4}{w}\Bigg[&\sum_{n=0}^{\infty} \cos^2\frac{(2n+1)\pi x}{w} \times \\ &\frac{\sinh^2[(n+1/2)\pi(h+2y)/w]}{\sinh^2[(2n+1)\pi h/w]} + \\ &+ \sum_{n=1}^{\infty} \sin^2\frac{2n\pi x}{w}\,\frac{\sinh^2[n\pi(h+2y)/w]}{\sinh^2[2n\pi h/w]}\Bigg] \\ &+ \{x \leftrightarrow y;\; w \leftrightarrow h\}\,. \end{aligned} \tag{9}$$

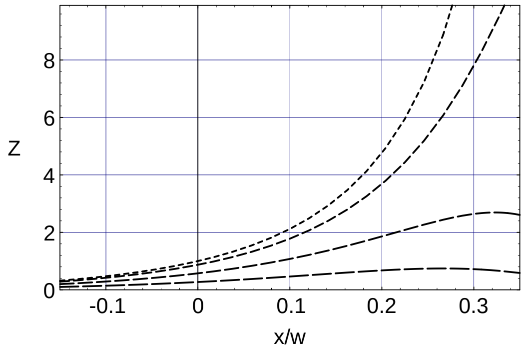

Figure 2: Impedance of a hole in the middle of square-pipe wall ($y_h/h = 1/2$) versus horizontal beam offset for different vertical beam offsets $y/h = 0$ (short-dashed), 0.1, 0.2 and 0.3 (long-dashed). For an on-axis beam $Z = 1$.

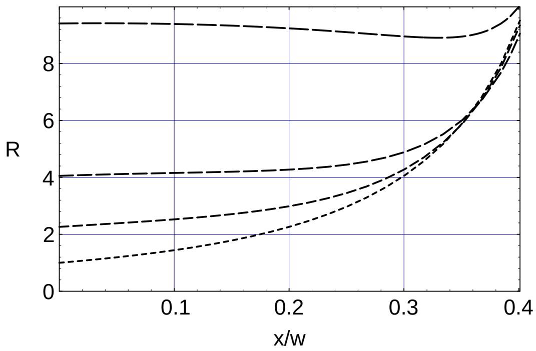

Figure 3: Power loss in square pipe versus horizontal beam offset for different vertical beam offsets $y/h = 0$ (no offset, short-dashed), 0.2, 0.3, and 0.4 (long-dashed). $R = 1$ corresponds to an on-axis beam.

An example of a square pipe is illustrated in Fig. 3.

For a centered beam, i.e. $x = y = 0$, it reduces to

$$\oint_{\partial S} dl\; e_\nu^2(0) = \frac{1}{w}\sum_{n=0}^{\infty}\cosh^{-2}\frac{(2n+1)\pi h}{2w} + \{w \leftrightarrow h\}\,, \tag{10}$$

the result obtained in [4], which was also expressed in a closed form in terms of elliptic integrals [8].

4 ON TRANSVERSE IMPEDANCE

The longitudinal and transverse wake functions are related by Panofsky-Wenzel theorem

$$\vec{\nabla}\,W(z, \vec{a}\,) = \frac{\partial}{\partial z}\vec{W}_\perp(z, \vec{a}\,)\,. \tag{11}$$

The longitudinal wake function corresponding to the inductive impedance (1) of the hole is $W(z,\vec{a}\,) = \delta'(z)F(\vec{a}\,)$, where $F(\vec{a}\,) = Z_0 e_\nu^2(\vec{a}\,)(\psi-\chi)/2$. Together with Eq. (11), it implies $\vec{W}_\perp(z,\vec{a}\,) = \delta(z)\vec{\nabla}\,F(\vec{a}\,)$, and the monopole transverse impedance defined as the Fourier transform of $\vec{W}_\perp(z,\vec{a}\,)$ in $\tau = z/c$, is

$$\vec{Z}_\perp^{mon}(k,\vec{a}\,) = \frac{1}{c}\,\vec{\nabla}\,F(\vec{a}\,) = Z_0\frac{\psi-\chi}{2}\vec{\nabla}\,e_\nu^2(\vec{a}\,)\,. \quad (12)$$

Defined in such a way $\vec{Z}_\perp^{mon}$ has dimension of Ohms, and can be easily calculated when $e_\nu(\vec{a}\,)$ is found, e.g. Eqs. (6) or (8). In an axisymmetric pipe, $\vec{Z}_\perp^{mon} = 0$, e.g. [1], which formally follows from the fact that Z_{long} is independent of the beam position in such a case. However, presence of a hole breaks this symmetry, so that $\vec{Z}_\perp^{mon}$ does not vanish even on the axis. For example, for a circular chamber with a hole

$$\vec{Z}_\perp^{mon}(k,0) = Z_0\frac{\psi-\chi}{4\pi^2 b^3}\vec{h}\ , \quad (13)$$

where $\vec{h}$ is a unit vector from the axis toward the hole. The presence of a second, symmetric hole (or a few of them) restores the symmetry, and this effect disappears.

The transverse kick obtained by a test charge q_t which follows, at distance $z \geq 0$, the leading charge q_s, is

$$\vec{p}_\perp(z,\vec{a}\,) = \frac{q_t q_s}{c}\vec{W}_\perp(z,\vec{a}\,) = \frac{q_t q_s}{c}\delta(z)\vec{\nabla}\,F(\vec{a}\,)\,. \quad (14)$$

As an example, Fig. 4 shows the direction and magnitude of the monopole impedance and corresponding transverse kick in a circular pipe. For a rectangular chamber, the picture is similar. The result (14) looks suspicious due to $\delta(z)$, which means there is no influence on any test charge with $z > 0$, while self-influence of the source charge diverges. One should attribute this unphysical behavior to the approximations used: (i) point-like discontinuity, (ii) ultrarelativistic charge, and (iii) instant induction of effective dipoles on the hole. A rigorous approach, taking into account $\beta < 1$ and a finite hole size, would lead to a more appropriate longitudinal dependence, although calculations will be certainly complicated. An involved direct calculation (using the method of the second paper of Ref. [2], again with $\beta = 1$) of the integrated transverse force acting on an on-axis charge passing a hole in a circular pipe leads to divergent sums which, however, would be natural to put equal to zero[1]. Anyway, this question remains open.

The more usual dipole transverse coupling impedance in the chamber with a hole, e.g. [3, 9, 6], reflects the influence of a couple of opposite-charged particles with transverse offsets $(\vec{s}, -\vec{s})$ on a test charge with offset $\vec{t}$:

$$\vec{Z}_\perp^{dip}(k,\vec{s},\vec{t}\,) = -iZ_0\frac{\psi-\chi}{2}\,\frac{e_\nu(\vec{s}\,)-e_\nu(-\vec{s}\,)}{2s}\,\vec{\nabla}e_\nu(\vec{t}\,)\ , \quad (15)$$

where the limit $\vec{s} \to \vec{t} \to 0$ is usually assumed. If instead one considers $\vec{t} \to 0$ while keeping $\vec{s} = \vec{a}$ finite, we get corrections to the transverse dipole impedance. For example,

[1] Remark due to R.L. Gluckstern

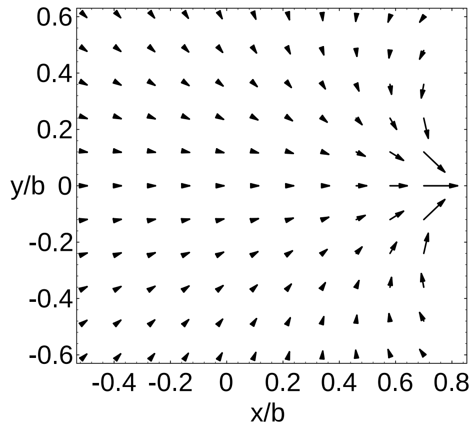

Figure 4: Direction and magnitude of monopole transverse impedance in central region of circular pipe with a hole at $\varphi_h = 0$ ($x = b$, $y = 0$) versus beam position, normalized to that magnitude for an on-axis beam.

in a circular pipe ($\varepsilon = a/b < 1$)

$$\vec{Z}_\perp^{dip}(k,\vec{a}) = -iZ_0\frac{\psi-\chi}{2\pi^2 b^4}\,\vec{h}\,\cos(\varphi_a-\varphi_h)\times \quad (16)$$
$$\frac{1-\varepsilon^2}{(1+\varepsilon^2\,)^2 - 4\varepsilon^2\cos^2(\varphi_a-\varphi_h)}\,.$$

In the limit of $a \to 0$ it reproduces the known result for the transverse dipole impedance of the hole, the first line in (16), cf. [2, 3]. It corresponds to the deflecting force directed toward (or opposite to) the hole with its magnitude proportional to the beam offset and depending on beam azimuth position φ_a as $\cos(\varphi_a - \varphi_h)$. Expanding in powers of ε yields sextupole term and higher-order corrections:

$$\cos\left(\varphi_a-\varphi_h\right) + \varepsilon^2\cos 3(\varphi_a-\varphi_h) + O\left(\varepsilon^4\right)\ .$$

Results for rectangular pipes are obtained in a similar way from Eq. (8) in terms of series.

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