On RMS Envelope Equation Problem for Nonlinear Motion

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Abstract

The problem of the model RMS envelope equation construction for a nonlinear motion is been studied. The RMS equation is deduced from an infinite system of linear differential equations for moments with the help of Bogoljubov's asymptotic method. For example, an ensemble of nonlinear oscillators is considered in external stationary fields (smooth channel).

1 INTRODUCTION

The RMS phase space dimensions of the beam are defined through the matrix of the second order central moments of the particle distribution function M_{ik}. In linear electromagnetic fields the time evolution of the moment matrix is described by the matrix equation

$$\frac{dM}{dt} = MA + \tilde{A}M, \tag{1}$$

where sign $\sim$ defines a transposed matrix, A is a matrix that define a particle motion in external fields [1].

The infinite chain of the linear differential equations for the moments of different orders corresponds to the particle motion in nonlinear fields. So the equation for the second order moments has the form

$$\frac{dM}{dt} = MA + \tilde{A}M +$$

$$+B_2(\, a\, linear\; function\; of\; high\; order\; moments). \tag{2}$$

The equations for the high order moments have the similar form. For linear fields the infinite chain is subdivided into systems of equations for the moments of the same order. To obtain a solution for a small nonlinearity it is naturally to cut off the equation chain by means of omitting the moments of the chosen high order. Really such procedure may be successful for the simulation of a beam transportation through a channel of the limited length (for example there is RMS envelope equation using the high order moment calculations [2]). On the other hand for the periodic structure or during the time the secular ($\sim t^n$) terms must occur in the solution similar to well known ones in the perturbation theory. Therefore a special procedure is needed to investigate the time evolution of an ensemble of nonlinear oscillators in the external spatial periodical fields or a beam transportation in the long channels. One such approach is treated in this paper.

2 FORMAL SOLUTION OF MOMENT EQUATIONS

For the beginning let us find the general solution of the matrix equation (2) for the ensemble of nonlinear oscillators. The solutions for the high order moment equations would be obtained in a similar way.

Let us represent a second order moment symmetrical matrix M in a vector form by a column (or a row for transposed one) $\tilde{M} = (M^{2,0}, M^{1,1}, M^{0,2})$ and rewrite (2) in the corresponding form

$$\frac{dM}{dt} = AM + B, \tag{3}$$

where matrices A and B correspond to the former A and B_2 but not coincide with them. For the linear oscillations ($B = 0$)with a frequency $\omega = 1$ the equation (3) has three linear independent solutions

$$M_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, M_1 = \begin{pmatrix} cos\tau \\ -sin\tau \\ -cos\tau \end{pmatrix}, M_2 = \begin{pmatrix} sin\tau \\ cos\tau \\ -sin\tau \end{pmatrix},$$

where $\tau = 2t$. On the base of these solutions one can find the Green's function corresponding to the equation (3)

$$G(\tau, \tau') = M_0(\tau)\tilde{M}_0^*(\tau') + M_1(\tau)\tilde{M}_1^*(\tau') + M_2(\tau)\tilde{M}_2^*(\tau'), \tag{4}$$

where the vectors M_i and M_k^* are mutualy ortogonal and normalized vectors

$$\tilde{M}_i^*(\tau)M_k(\tau) = \tilde{M}_i(\tau)M_k^*(\tau) = \delta_{ik}. \tag{5}$$

The general solution of the equation (3) is given by

$$M = c_0(\tau)M_0 + c_1(\tau)M_1 + c_2(\tau)M_2, \tag{6}$$

where

$$c_i(\tau) = c_i(0) + \frac{1}{2}\int_0^\tau \tilde{M}_i^*(\tau')B(\tau')\, d\tau'. \tag{7}$$

In general case $B(\tau)$ has a structure

$$B(\tau) = \sum_{N=2}^{\infty} P_{2N}^2(\tau)M^{2N}(\tau), \tag{8}$$

where $P_{2N}^2(\tau)$ is a $3 \times (2N+1)$ matrix with elements defined by a nonlinear part of a force, and $M^{2N}(\tau)$ are columnes corresponding to the moment matrices of order $2N$.

For the moments of order $2N$ we obtain in a similar way

$$M^{2N} = \sum_{i=0}^{2N} c_i^{2N}(\tau) M_i^{2N}(\tau), \qquad (9)$$

where $M_i^{2N}(\tau)$ are eigenvectors of a linear problem similar to those for second order moments, and

$$c_i^{2N}(\tau) = c_i^{2N}(0) + \frac{1}{2}\int_0^\tau \tilde{M}_i^{*2N}(\tau') B_{2N}(\tau')\, d\tau', \quad (10)$$

$$B_{2N}(\tau) = \sum_{2k=2N+2}^{\infty} P_{2k}^{2N}(\tau) M^{2k}(\tau), \qquad (11)$$

$\tilde{M}_i^{*2N}$ are defined by the equations similar to (5).

If we consider the small nonlinearity, i. e. $P_{2k}^{2N} \sim \varepsilon$ and $M^{2k}(\tau) = M_l^{2k}(\tau) + \varepsilon \Delta M^{2k}(\tau)$, where $M_L^{2k}(\tau)$ is a linear part of these moments, than in the first order approximation we have

$$c_i^{2N}(\tau) = c_i^{2N}(0) +$$
$$+\frac{1}{2}\sum_{2k=2N+2}^{\infty}\int_0^\tau \tilde{M}_i^{*2N}(\tau') P_{2k}^{2N}(\tau) M_L^{2k}(\tau')\, d\tau', \qquad (12)$$

that is valid for the time intervals defined as $\varepsilon\tau \ll 1$. One can see from this formula that used cut of the moment equation chain leads to the secular term appearance in a general case. Nevertheless this method may be useful especially for non-periodical channels with a long drift sections.

3 OSCILLATORS IN STATIONARY EXTERNAL FIELDS.

Let us consider the ensemble of oscillators in stationary external fields. It means that P_{2k}^{2N} does not depend on time.

For $i \neq 0$ the vectors M_i^{2N} can be divided on the pairs corresponding to the same frequency (we can put if $i = 2k+1$, than the vectors M_i^{2N} and M_{i+1}^{2N} have the same frequency $i = 2k+1$). The summation formulae are valid for the vectors $M_{2k+1^{2N}}$ and M_{2k+2}^{2N}

$$M_{2k+1}^{2N}(\tau+\varphi) = \cos{(2k+1)\varphi} M_{2k+1}^{2N}(\tau) -$$
$$-\sin{(2k+1)\varphi} M_{2k+2}^{2N}(\tau); \qquad (13)$$
$$M_{2k+2}^{2N}(\tau+\varphi) = \cos{(2k+1)\varphi} M_{2k+2}^{2N}(\tau) +$$
$$+\sin{(2k+1)\varphi} M_{2k+1}^{2N}(\tau). \qquad (14)$$

The similar equations are fulfilled for the vectors M_i^{*2N}

$$\tilde{M}_{2k+1}^{*2N}(\tau+\varphi) = \cos{(2k+1)\tau} \tilde{M}_{2k+1}^{*2N}(\varphi) -$$
$$-\sin{(2k+1)\tau} \tilde{M}_{2k+2}^{*2N}(\varphi); \qquad (15)$$
$$\tilde{M}_{2k+2}^{*2N}(\tau+\varphi) = \cos{(2k+1)\tau} \tilde{M}_{2k+2}^{*2N}(\varphi) +$$
$$+\sin{(2k+1)\tau} \tilde{M}_{2k+1}^{*2N}(\varphi); \qquad (16)$$

The linear part of the moment vector $M_L^{2N}(\tau)$ can be represented as a linear combination of the vectors $M_i^{2N}(\tau)$

$$M_L^{2N}(\tau) = \sum_{k=0}^{N-1} A_k[\cos\varphi_k M_{2k+1}^{2N}(\tau) - \sin\varphi_k M_{2k+2}^{2N}(\tau)] +$$
$$+M_{L,0}^{2N}, \qquad (17)$$

where $M_{L,0}^{2N}$ is a time indepentent part of M_L^{2N} and A_k, φ_k are arbitrary constants. In the stationary case the vector $M_{L,0}^{2N}$ must satisfy the equation

$$\tilde{M}_0^{*2N} P_{2k}^{2N} M_{L,0}^{2k} = 0 \qquad (18)$$

Using equations (12- 17) one can see the secular terms appearance for the chosen approximation. The slow change of the coefficient $c_i^{2N}(\tau)$ we can get by the averaging (12) over the fast oscilations

$$c_i^{2N}(\tau) = c_i^{2N}(0) +$$
$$+\frac{1}{2}\sum_{2k=2N+2}^{\infty}\int_0^\tau \overline{\tilde{M}_i^{*2N}(\tau') P_{2k}^{2N}(\tau) M_L^{2k}(\tau')}\, d\tau'. \qquad (19)$$

Using the method described above one can obtain the solution for the Duffing oscillator (small ($\sim \varepsilon$) cubic nonlinearity). In this case the moment equation chain separates for the moment of different parity and the equations for the even order moments correspond to the central ones. For the second order moments of the Duffing oscillator we have

$$c_1^{(2)}(\tau) = \frac{a^2}{2}(\cos\varphi_0 - \frac{3\varepsilon a^2}{4}\tau\sin\varphi_0) \qquad (20)$$

$$c_2^{(2)}(\tau) = -\frac{a^2}{2}(\sin\varphi_0 + \frac{3\varepsilon a^2}{4}\tau\cos\varphi_0) \qquad (21)$$

Following the Bogolubov's averaging methods we can put $\Delta\varphi(\tau) = \Delta\omega\tau$, $\Delta\omega = \frac{3\varepsilon a^2}{4}$, and $\sin\Delta\omega\tau = \frac{3\varepsilon a^2}{4}\tau + o(\varepsilon^3)$, $\cos\Delta\omega\tau = 1 + o(\varepsilon^2)$.

Then, using (13- 14), we obtain

$$M^{(2)} = \frac{a^2}{2}[M_{L,0}^{(2)} + M_1^{(2)}(\tau + \Delta\varphi(\tau)] \qquad (22)$$

in a good agreement with the direct calculations for one paticle. Evidently, the formula (22) is valid for an arbitrary particle motion in a stationary external field (with an odd spatial parity) or for the central second order moments in the case of arbitrary stationary external fields if all particles are distributed in a phase space layer $a = const$. There are the freaquency nonlinear shift that is equal for all particles in this case. Therefore there is not any RMS emittance time dependence.

The general solution of the moment dynamics problem is a sum of the solution as in (22) with different amplitudes and initial phases. For the finite number of layers and at the beginning of the motion there are only the RMS dimension beating. For the infinite layer numbers it is possible to try to construct a model with account of the RMS dimension damping due to phase mixing. The calculations show that the nonlinear frequency shift displays at time intervals $t \ll \dfrac{1}{\Delta\omega}$ and the oscillation damping as well as the emittance growth occurs with a significant delay $t \sim \dfrac{1}{\Delta\omega}$.

4 MODEL RMS ENVELOPE EQUATION

The well-known RMS envelope equation [3], [4] and its later modifications are very useful for solving of various problems of the beam dynamics. So it seems attractive to have the similar equation for the nonlinear beam dynamics. On the base of the discussed results such RMS envelope equation for the stationary external fields must include three additional parameters in addition to the linear motion case. They are the nonlinear frequancy shift, the RMS oscillation damping time and the emittance growth time. To illustrate the construction of such sort equation we consider the simplest case of an ensemble of Duffing oscillators. For the second order moments we have the equations

$$\frac{da^2}{dt} = 2 < xv >; \quad \frac{d < xv >}{dt} = < v^2 > -\omega^2 a^2 - D^{4,0};$$

$$\frac{d < v^2 >}{dt} = -2 < xv > - \frac{dD^{4,0}}{dt}, \tag{23}$$

where $< .. >$ shows the averaging on an ensemble here and $D^{4,0} = \varepsilon < x^4 >$. The slow RMS emittance (F) time dependence is defined by

$$\frac{d\overline{F^2}}{dt} = \overline{D^{4,0}\frac{da^2}{dt} - a^2\frac{dD^{4,0}}{dt}} \tag{24}$$

The analysis of a time behavior of $D^{4,0}$ with the help of the Bogolubov's first approximation shows that we can approximate

$$\overline{F^2} = (F_f^2 - F_i^2)f(t) + F_i^2, \tag{25}$$

$f(t)$-is an arbitrary increasing function with the values $f(0) = 0$, $f(t) \to 1)$ when $t \gg \tau_0$ and τ_0 is a mixing time parameter; F_f^2, F_i^2 are the final and initial values of F^2. The value of F_f^2 can be obtained with account of the ergodic theorem and the energy conservation law.

Besides we can introduce an approximation

$$D^{4,0} = \Delta\omega^2 a^2 + \frac{p}{\tau_0}\frac{da^2}{dt}, \tag{26}$$

where $\Delta\omega^2$ is a parameter corresponding to the initial nonlinear frequency shift and $p \sim 1$. Using this approximations and the equations (23) we obtain the RMS envelope equation

$$\frac{d^2a}{d^2t} + (\omega^2 + \Delta\omega^2)a + \frac{p}{\tau_0}\frac{da}{dt} - \frac{\overline{F^2}}{a^3} = 0. \tag{27}$$

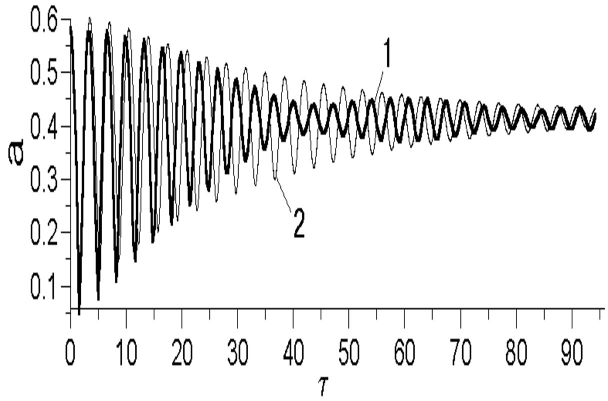

Figure 1: The evolution of the RMS dimension of the Duffing oscillator ensemble

The RMS envelope equations (25, 27) have a general form and therefore are applicable for arbitrary conservative nonlinear fields. For example, the results of calculations of RMS dimensions by the macroparticle method (curves 1) and with the help of the equations (25, 27) (curves 2) are shown in Fig. 1. The time evolution of Duffing oscillators with the finite linear initial distribution in the phase space were chosen for the illustration. The averaged emittance square was approximated by the exponential function.

5 REFERENCES

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