DIFFRACTION MODEL OF A STEP-OUT TRANSITION *

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Abstract

The diffraction model of a cavity, suggested by Lawson [1], Bane and Sands [2] is generalized to a step-out transition. Using this model, the high-frequency impedance is calculated explicitly for the case that the transition step is small compared with the beam-pipe radius. In the diffraction model for a small step-out transition, the total energy is conserved, but, unlike the cavity case, the diffracted waves in the geometric shadow and the pipe region, in general, do not always carry equal energy. In the limit of small step sizes, the impedance derived from the diffraction model agrees with that found by Balakin, Novokhatsky [3] and also Kheifets [4]. This impedance can be used to compute the wake field of a round collimator whose half-aperture is much larger than the bunch length, as existing in the SLC final focus.

1 SHORT CAVITY (REVIEW)

The high-frequency impedance of a cylindrically symmetric short cavity structure in an otherwise smooth vacuum chamber pipe can be estimated by a diffraction model as described in Refs. [1, 2]. It is helpful to first review this model, before generalizing it to a step-out transition.

Let b be the radius of the smooth beam pipe, g the cavity gap length, and d its depth. A schematic is shown in Fig. 1. As a beam current $J_0 = \hat{J}_0 \exp(-i\omega(t - s/c))$ enters the

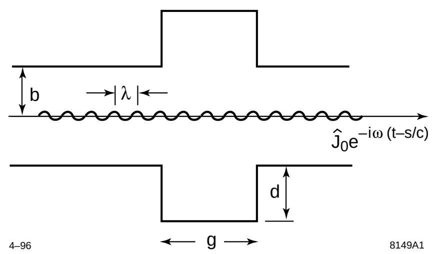

Figure 1: Schematic view of a short cavity

cavity along the beam pipe axis, a diffracted wave is generated at the cavity edge $r = b$. This diffracted wave propagates down the beam pipe while spreading out radially due to diffraction. At a longitudinal distance s behind the entrance edge of the cavity, the radial spread of the diffracted wave is about

$$\Delta y(s) \approx \frac{1}{2\pi}\sqrt{\frac{\lambda s}{2}} \tag{1}$$

where $\lambda = 2\pi c/\omega$. In the diffraction model of Refs. [1, 2, 5], the cavity gap length g is assumed to be short enough that the wave has not spread out radially to reach the outer wall of the cavity. Thus the outer cavity wall does not play a role in determining the short range wake field, and the quantity d does not enter the considerations.

In calculating the impedance of the above cavity structure, the wavelength λ is assumed to be sufficiently short that the diffracted wave populates only the radial region close to $r = b$, and, in particular, neither penetrates much into the depth of the cavity structure nor approaches the pipe axis. In this case, one can approximate the cylindrical geometry near the $r = b$ region by a Cartesian geometry and represent the incoming beam wave by a planar wave with $E_y = -B_x = -2J_0/(cb)$. The monopole longitudinal impedance is then found to be [2, 5, 6]

$$Z_0^{\parallel}(\omega) = [1 + \mathrm{sgn}(\omega)\ i]\frac{Z_0}{2\pi^{3/2}}\frac{1}{b}\sqrt{\frac{cg}{|\omega|}} \tag{2}$$

where $Z_0 = 377\ \Omega$. The $m \neq 0$ longitudinal and transverse impedances $Z_m^{\parallel}$ and $Z_m^{\perp}$ are obtained by considering an mth moment current $J_m = \hat{J}_m \exp(-i\omega(t - s/c))$. The corresponding planar wave incident upon the cavity entrance is $E_y = -B_x = -\frac{4}{cb^{m+1}} J_m \cos m\theta$, where θ is the azimuth at the cavity entrance. The final result is [2, 5]

$$Z_m^{\parallel}(\omega) = [1 + \mathrm{sgn}(\omega)\ i]\frac{Z_0}{\pi^{3/2}}\frac{1}{b^{2m+1}}\sqrt{\frac{cg}{|\omega|}} \tag{3}$$

The transverse impedance follows from the Panofsky-Wenzel theorem, $Z_m^{\perp}(\omega) = cZ_m^{\parallel}(\omega)/\omega$. It has been shown that exactly half of the diffracted wave energy is contained in the geometric shadow region (outward diffracted), while the other half is contained in the region propagating down the pipe (inward diffracted) [2, 5].

2 STEP-OUT TRANSITION

Equations (2) and (3) apply at high frequencies and short cavity lengths, when $d \gg \sqrt{\lambda g/2}/(2\pi)$. The purpose of this paper is to extend the 'conventional' diffraction model just described to the case when the cavity gap length g is long, and, thus, the above condition is violated. The cavity structure in this case resembles a transition step in the vacuum chamber pipe. We further assume the step to be small, i.e., $b \gg d$, so that we can still approximate the cylindrical geometry by a planar one.

Figure 2(a) displays the geometry near $r = b$. Shown shaded is the region populated by the diffracted wave. When

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the latter reaches the outer radius after the step, it is reflected by the pipe wall. This reflection can be represented by an image beam current $-J_0$ as shown in Fig.2(b).

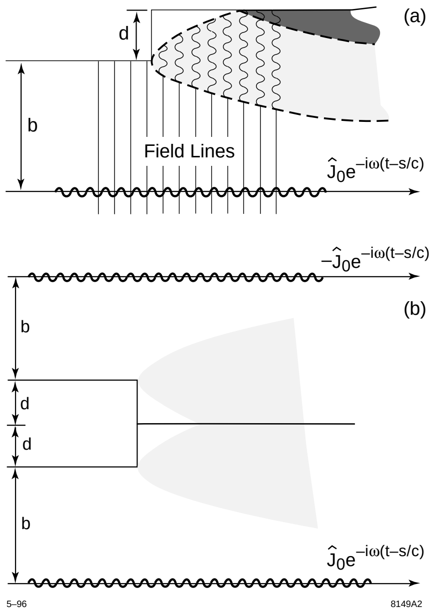

Figure 2: a) Diffracted waves near a step-out transition; b) equivalent geometry with image current.

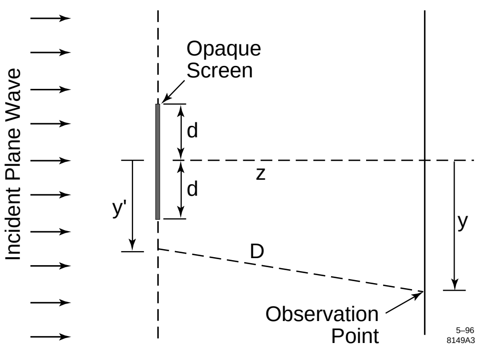

Figure 3: Diffraction of plane wave on a screen.

The diffracted wave is then the same as that for a plane wave incident perpendicularly upon a screen, which is illustrated in Fig. 3. The parts of the screen with $y' > d$ and $y' < -d$ are transparent to the incident wave, while the part with $-d < y' < d$ is opaque. The wave amplitude at an observation point located a longitudinal distance s behind the screen and at a transverse distance y from the pipe axis, is proportional to

$$a(y,s) \propto \left[\int_{-\infty}^{-d} dy' + \int_{d}^{\infty} dy'\right] e^{i\omega D/c} \tag{4}$$

where

$$D = \sqrt{s^2 + (y-y')^2} \approx s + \frac{1}{2s}(y-y')^2 \tag{5}$$

and we have adopted a scalar wave model of the electromagnetic waves [7] as was done in the 'conventional' diffraction model. Substituting Eq.(5) into Eq.(4) and dropping the oscillatory factor $\exp(i\omega s/c)$, we find

$$\begin{aligned} a(y,s) &\propto \left[\int_{-\infty}^{-d} dy' + \int_{d}^{\infty} dy'\right] \exp\left[\frac{i\omega}{2cs}(y-y')^2\right] \\ &= \sqrt{\frac{\lambda s}{2}} \Big\{ \left[1 - C\left(f_+(y,s)\right) + C\left(f_-(y,s)\right)\right] \\ &\qquad + i\left[1 - S\left(f_+(y,s)\right) + S\left(f_-(y,s)\right)\right] \Big\} \end{aligned}$$

where $f_\pm(y,s) \equiv \sqrt{\frac{2}{\lambda s}}(y \pm d)$, and, by definition of the Fresnel integrals,

$$C(x) + iS(x) \equiv \int_0^x dt\, e^{i\pi t^2/2} \tag{6}$$

with $C(\infty) = S(\infty) = 1/2$. The energy flux $F(y,s)$ is proportional to $|a(y,s)|^2$. By normalizing $F(y,s)$ such that for $y \to \infty$ it is equal to the incident flux

$$F(y,s) \;\to\; F_0 = \frac{c}{8\pi}(E_y^2 + B_x^2) = \frac{J_0^2}{\pi c b^2}, \tag{7}$$

we obtain

$$\begin{aligned} F(y,s) &= \frac{F_0}{2}\Big\{\left[1 - C(f_+(y,s)) + C(f_-(y,s))\right]^2 \\ &\quad + \left[1 - S(f_+(y,s)) + S(f_-(y,s))\right]^2\Big\}. \end{aligned} \tag{8}$$

The total field energy contained in the diffracted wave pattern is conserved [8], i.e., for all s, we have

$$\int_0^d dy F(y,s) + \int_d^\infty \left[F(y,s) - F_0\right] = 0. \tag{9}$$

In Fig. 4, the flux $F(y,s)/F_0$ is depicted as a function of y/d for several values of $\alpha \equiv d\sqrt{2/(\lambda s)}$. Before the diffracted wave reaches the outer cavity wall ($s \le 8\pi^2 d^2/\lambda$ or $\alpha \ge 1/(2\pi)$), the flux is about the same as in the conventional diffraction model. By contrast, for $s \ge 8\pi^2 d^2/\lambda$ or $\alpha \le 1/(2\pi)$, the reflection from the outer pipe wall becomes important. In this region the asymptotic expression for $F(y,s)$ reads

$$\frac{F(y,s)}{F_0} \approx 1 - 2\alpha\left[\cos\left(\frac{\pi}{\lambda s}y^2\right) + \sin\left(\frac{\pi}{\lambda s}y^2\right)\right] \tag{10}$$

The second term on the right-hand side of Eq.(10) is a small correction term, so that the energy flux far from the cavity entrance edge is approximately equal to the unperturbed value F_0. However, this does not mean the field energy in the inward diffracted wave diminishes as the wave propagates down the beam pipe, because the diffracted wave is spreading and is occupying larger volume as it propagates.

To derive the impedance, we first calculate the energy loss of the beam. This is equal to the field energy stored in the diffracted wave and, thus, it is proportional to

$$\int_0^d dy|a(y,s)|^2 + \int_d^\infty dy|a(y,s) - a(\infty,s)|^2 \tag{11}$$

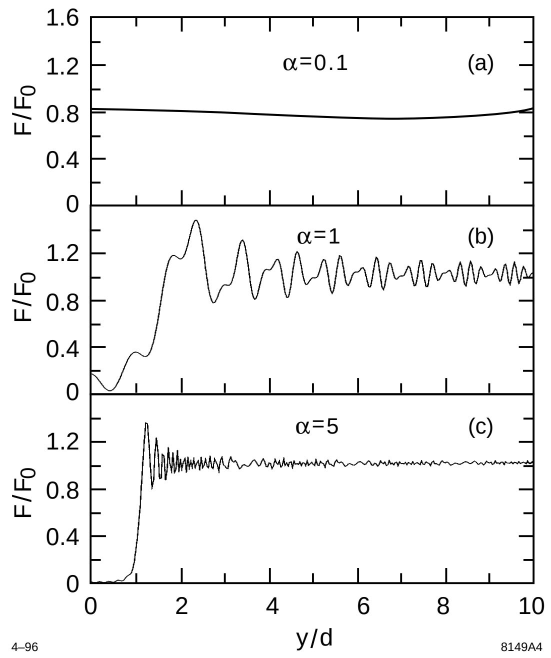

Figure 4: Energy flux F/F_0 as a function of transverse position at various distances $z \propto 1/\alpha^2$ behind the step-out.

The first term in Eq.(11) is proportional to the outward diffracted wave energy in the geometric shadow region and the second term represents the inward diffracted wave in the pipe region. When expanding in α (after multiplication with $\sqrt{2/(\lambda s)}$), the first term is found to be: $2\alpha - 4\alpha^2 + 4\alpha^3 + \mathcal{O}(\alpha^5)$; and the second is: $2\alpha - 4\alpha^3 + \mathcal{O}(\alpha^5)$ [8]. In the long-distance limit $\alpha \ll 1$, or $z \gg 8\pi^2 d^2/\lambda$, the two terms become equal, which means that sufficiently far away from the step the outward diffracted and the inward diffracted waves contain equal amounts of energy. The loss power for a small step size ($d \ll b$) is, therefore, given by

$$P_0 \approx 2 \times 2\pi b \lim_{s\to\infty} \int_0^d F(y,s)\, dy. \qquad (12)$$

Performing the integration for $s \gg 2d^2/\lambda$, one finds [8]:

$$P_0 \approx \frac{4J_0^2}{cb} d. \qquad (13)$$

With $F_0 \equiv J_0^2/(\pi b^2 c)$, the energy loss can be rewritten as $P_0 \approx 2\pi b(F_0 2d)$, which demonstrates that, in our model, the total diffracted-wave energy is equal to the energy incident on the screen of Fig. 3. This equivalence also follows from Babinet's principle [7], when one considers the inverse problem of a plane wave incident on a slit of width $2d$.

The loss power is related to the impedance via $P_0 = (\mathrm{Re} Z_0^{\parallel}) J_0^2$, from which

$$\mathrm{Re} Z_0^{\parallel} \approx \frac{Z_0}{\pi b} d. \qquad (14)$$

Unlike the short-cavity impedance of Eq. (2), the high-frequency impedance for a long cavity, or a step, is independent of frequency ω. Equation (14) agrees with the results of earlier treatments [3, 4] in the limit of small step sizes $d \ll b$. It is intriguing that the impedance of Eq. (14) can be obtained from the short-cavity impedance, Eq. (2), by replacing the cavity length g with the distance $8\pi^2 d^2/\lambda$ at which the diffracted wave reaches the outer beam-pipe radius after the step (see Eq. (1)).

In the same manner as for a short cavity, considering mth-moment currents allows the extension of the analysis to the $m \neq 0$ impedances. In this case the incident flux is

$$F_{m,0} = \frac{4}{\pi c b^{2m+2}} J_m^2 \cos^2(m\theta), \qquad (15)$$

and the loss power is obtained by integrating the diffracted flux over the azimuth θ with the final result:

$$P_m \approx \frac{8d}{(1+\delta_{0m})\, c b^{2m+1}} J_m^2 \qquad (16)$$

where δ_{0m} denotes the Kronecker delta ($\delta_{0m} = 0$ for $m \neq 0$). Equating P_m with the general expression $(\mathrm{Re} Z_m^{\parallel}) J_m^2$ yields the impedance for the mth-moment

$$\mathrm{Re} Z_m^{\parallel} \approx \frac{2 Z_0 d}{(1+\delta_{0m})\, \pi b^{2m+1}}. \qquad (17)$$

For small step sizes d, the dipole impedance (m=1) so obtained agrees with that derived by an entirely different approach in Ref. [9]. This impedance can be used, for example, to compute transverse collimator wake fields when the apertures are larger than the bunch length, as is the case in the SLC final focus [10].

3 REFERENCES

[1] J.D. Lawson, Rutherford Lab. Report RHEL/M 144 (1968), Part. Accel. 25, 107 (1990).

[2] K. Bane and M. Sands, Part. Accel. 25, 73 (1990).

[3] V.E. Balakin and A.V. Novokhatski, Proc. 12-th Int. Conf. High Energy Accel., Batavia, 1984, p.117.

[4] S.A. Kheifets, IEEE Trans. Microwave Theory Technique MTT-35, 753-760 (1987).

[5] Alexander W. Chao, Physics of Collective Beam Instabilities in High Energy Accelerators, J. Wiley, New York, 1993.

[6] A. Chao and H. Henke, SLAC-PUB-95-6780, Part. Accel. Conf., Dallas, 1995.

[7] J.D. Jackson, Classical Electrodynamics, John Wiley, New York, 1962; M. Born and E. Wolf, Principles of Optics, Pergamon Press, Oxford, 1980.

[8] A. Chao and F. Zimmermann, unpublished (1995).

[9] E. Gianfelice and L. Palumbo, 'Dipole Impedance of a Conducting Cylindrical Pipe with a Discontinuous Cross Section', IEEE Trans. N.S., Vol. 37, 2, 1084 (1990).

[10] K. Bane, C. Ng, F. Zimmermann, these proceedings.