Stability Analysis of a Beam Loaded Double R.F. System

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Abstract

The CERN PS Booster[2] uses a double r.f. system to lengthen the bunches and reduce the transverse space-charge tune shift. We present stability criteria for such a system under conditions of strong beam loading. The condition that all normal modes be exponentially damped, places constraints on the coefficients of the system characteristic polynomial developed in the Laplace frequency variable. For a sextic polynomial, as occurs with a double r.f. system, interacting with a single beam mode and no feedbacks, there are 7 Routh conditions and these are elucidated and interpreted in this paper.

1 INTRODUCTION

Double r.f. systems[1] promote longitudinal Landau damping and lengthen bunches so reducing the influence of space-charge. Both these effects are beneficial in the PS Booster[2] where an $h_1 = 5,\ h_2 = 10$ system is installed and $h_1 = 1,\ h_2 = 2$ is intended for LHC. Inevitably, the complication of the control system becomes and the possibility of beam-load related instability is doubled. The analysis of Wang[3] for the SSC-LEB fails to distinguish between single-particle and centroid motions and does not include the beam-phase loop; the present analysis considers both these issues.

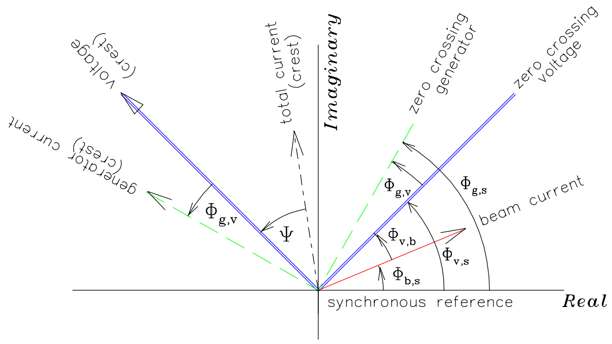

Figure 1: Phasor relation between voltage and currents.

By expanding the r.f. cavity space and time dependence as Fourier series, one may consider the charged particle to interact in approximate synchronism with a system of travelling waves. We assume sinusoidal carrier waves for the harmonic components, varying as $\mathcal{R}[\mathbf{A}_n(t)e^{+nj\omega t}]$. Here $\mathcal{R}$ is an instruction to take the real part of a complex quantity. We shall consider small modulations of the voltage, but large oscillations of the ion phase. Let us measure phases with respect to the synchronous reference particle, as in figure 1. Let the components of cavity-gap voltage, generator current and beam current be, respectively,

$$\mathbf{V}_n(t) \;=\; \mathbf{V}_n^0(1+\mathbf{e}_{vn}) \;\;;\;\; \mathbf{e}_{vn} = (a_{vn} + j\phi_{vn,s}) \quad (1)$$

$$\mathbf{I}_{gn}(t) \;=\; \mathbf{I}_{gn}^0(1+\mathbf{e}_{gn}) \;\;;\;\; \mathbf{e}_{gn} = (a_{gn} + j\phi_{gn,s}) \quad (2)$$

$$\mathbf{I}_{bn}(t) \;=\; \mathbf{I}_{bn}^0(1+\mathbf{e}_{bn}) \;\;;\;\; \mathbf{e}_{bn} = (a_{bn} + j\phi_{bn,s}) \quad (3)$$

Here a_n and ϕ_n are amplitude and phase modulations, respectively. The superfix 0 denotes steady state value; and the subscript n indicates which harmonic component. Let the infintessimal 'current' components due to an individual charge be

$$\delta\mathbf{I}_n = e^{nj\phi_{i,s}} \;. \quad (4)$$

1.1 Beam equation

Specifically for double harmonic operation, the phase (measured at the fundamental) of an single particle compared with the synchronous particle satisfies:

$$\ddot{\phi}_{i,s} + (\Omega_s^2/V_1^0)\mathcal{R}[\mathbf{V}_1\delta\mathbf{I}_1^* + \mathbf{V}_2\delta\mathbf{I}_2^* - (\mathbf{V}_1^0+\mathbf{V}_2^0)] = 0 \;. \quad (5)$$

Here superfix * indicates complex conjugate, and the synchrotron frequency is given by

$$\Omega_s^2 = -\frac{h_1\eta_s}{2\pi}\frac{c^2}{R_s^2}\frac{qV_1^0}{E_s} \;\;;\;\; \eta_s = \left[\alpha_p - \frac{1}{\gamma_s^2}\right] < 0 \;. \quad (6)$$

R_s = average ring radius and ρ_s = bending radius. The equilibrium solution is $\phi_{i,s} \equiv 0$. Hence the synchronous energy gain is:

$$\Delta E_s/q = 2\pi\rho_s R_s \dot{B}_s = -\mathcal{R}[\mathbf{V}_1^0 + \mathbf{V}_2^0] \;. \quad (7)$$

It is customary to take

$$\mathbf{V}_1^0 = V_1^0 e^{j(\Phi_{v1,s}+\pi/2)} \;\;;\;\; \mathbf{V}_2^0 = V_2^0 e^{j(\Phi_{v2,s}+3\pi/2)} \;. \quad (8)$$

For bunch-lengthening operation, one takes the first and second derivatives of the restoring force (w.r.t. $\phi_{i,s}$) to be zero (evaluated at $\phi_{i,s} = 0$) – though this is not essential.

$$V_1^0\cos\Phi_{v1,s} = \frac{h_2}{h_1}V_2^0\cos\Phi_{v2,s} \;;\; V_1^0\sin\Phi_{v1,s} = \frac{h_2^2}{h_1^2}V_2^0\sin\Phi_{v2,s} \quad (9)$$

Given V_1^0, the equations (7, 9) are enough to determine the relative amplitude and phase of the harmonic components.

To find the equation of motion for coherent oscillations, we must form the ensemble average of (5): $-\langle\ddot{\phi}_{i,s}\rangle =$

$$(\Omega_s^2/V_1^0)\mathcal{R}[\mathbf{V}_1\mathbf{I}_{b1}^*/(2I_{b0}) + \mathbf{V}_2\mathbf{I}_{b2}^*/(2I_{b0}) - (\mathbf{V}_1^0+\mathbf{V}_2^0)] \;. \quad (10)$$

where the beam current components are

$$\mathbf{I}_{bn} = 2I_{b0}\int_{-\pi}^{+\pi}\Lambda(\phi_{i,s})e^{nj\phi_{i,s}}d\phi_{i,s} \;. \quad (11)$$

Here I_{b0} is the d.c. component of the beam current, Λ is the bunch shape. It is customary to take $\mathbf{I}_{bn}^0 = I_{bn}^0 e^{j\Phi_{bn,s}}$ as the steady state, and this must satisfy the relation:

$$0 = \mathcal{R}[\mathbf{V}_1^0(\mathbf{I}_{b1}^0)^* + \mathbf{V}_2^0(\mathbf{I}_{b2}^0)^* - (\mathbf{V}_1^0+\mathbf{V}_2^0)2I_{b0}] \;. \quad (12)$$

Now, let us consider small beam modulations about the equilibrium; to first order we have:

$$0 = 2I_{b0}\ddot{\phi}_{b1,s} + (\Omega_s^2/V_1^0)\mathcal{R} \times [\mathbf{V}_1^0(\mathbf{I}_{b1}^0)^*(\mathbf{e}_{v1} + \mathbf{e}_{b1}^*) + \mathbf{V}_2^0(\mathbf{I}_{b2}^0)^*(\mathbf{e}_{v2} + \mathbf{e}_{b2}^*)] \ . \quad (13)$$

To simplify matters, consider the case that there is no amplitude modulation of the beam current (i.e. $a_{bn} \equiv 0$). Further let us suppose a rigid mode in which $\phi_{b2,s} = 2\phi_{b1,s}$.

1.2 Cavity equation

We now give the equations governing the steady state voltage and current in the cavity:

$$\mathbf{Y}_n\mathbf{V}_n^0 = \mathbf{I}_{gn}^0 + \mathbf{I}_{bn}^0] \ , \quad \mathbf{Y}_n = [1 - j\tan\Psi_n]/R_n \quad (14)$$

where $\mathbf{Y}_n$ is the admittance and, from detuning, $\tan\Psi_n = Q_n(\omega_{cn}^2 - n^2\omega^2)/(n\omega\omega_{cn})$. Here ω_{cn} is the cavity resonance frequency for the nth r.f. component, and R_n and Q_n are the shunt resistance and quality factor of that resonance, respectively. It is convenient to take:

$$\mathbf{I}_{g1}^0 = I_{g1}e^{j(\Phi_{g1,s}+\pi/2)} \ ; \ \mathbf{I}_{g2}^0 = I_{g2}e^{j(\Phi_{g2,s}+3\pi/2)} \ . \quad (15)$$

We may take real and imaginary parts to find the equilibrium conditions. Often the cavities are operated as a matched resistive load so that generator current and gap voltage are inphase, i.e. $\Phi_{gn,vn} \equiv 0$.

The next step is to find the evolution of small modulations about the steady state. Provided that the modulation frequency is much smaller than the carrier and that the cavity quality factor is large enough (i.e. $Q > 100$, say) then the time varying voltage and current obey:

$$[(\tau_{cn}/R_n)(d/dt) + \mathbf{Y}_n]\mathbf{V_n} = \mathbf{I}_{gn} + \mathbf{I}_{bn} \ . \quad (16)$$

Here $\tau_{cn} = 2Q_n/\omega_{cn}$ is the time constant of the cavity resonance. We subtract the steady state relation (14) to find the modulations:

$$\mathbf{V}_n^0[(\tau_{cn}/R_n)(d/dt) + \mathbf{Y}_n]\mathbf{e}_{vn} = \mathbf{I}_{gn}^0\mathbf{e}_{gn} + \mathbf{I}_{bn}^0\mathbf{e}_{bn} \quad (17)$$

We may compare real and imaginary parts to obtain the equations of motion for a_{vn} and $\phi_{vn,s}$.

2 NO CONTROL LOOPS

For the stability analysis, we form the Laplace transform of the beam and cavity equations w.r.t. the complex frequency s, so as to give a system of algebraic equations.

The state vector is $(a_{v1}, \phi_{v1,s}, \phi_{b1,s}, a_{v2}, \phi_{v2,s})$.
The determinantal matrix

$$\begin{vmatrix} 1+s\tau_1 & \tan\Psi_1 & -Y_{b1}\cos\Phi_{b1} & 0 & 0 \\ -\tan\Psi_1 & 1+s\tau_1 & +Y_{b1}\sin\Phi_{b1} & 0 & 0 \\ -A\sin\Phi_{b1} & -A\cos\Phi_{b1} & s^2+\Delta & B\sin\Phi_{b2} & B\cos\Phi_{b2} \\ 0 & 0 & +2Y_{b2}\cos\Phi_{b2} & 1+s\tau_2 & \tan\Psi_2 \\ 0 & 0 & -2Y_{b2}\sin\Phi_{b2} & -\tan\Psi_2 & 1+s\tau_2 \end{vmatrix} = 0 \quad (18)$$

Here $\Delta = A\cos\Phi_{b1} - 2B\cos\Phi_{b2}$ where $A = \Omega_s^2 I_{b1}/(2I_{b0})$ and $B = \Omega_s^2(V_2^0/V_1^0)(I_{b2}/2I_{b0})$ and $\Phi_{b2} \equiv \Phi_{v2,b2}$ and $\Phi_{b1} \equiv \Phi_{v1,b1}$. We introduce dimensionless current ratios $Y_{gn} = I_{gn}^0/I_{vn}^0$ and $Y_{bn} = I_{bn}^0/I_{vn}^0$ where $I_{vn}^0 = V_n^0/R_n$.

2.1 Characteristic polynomial

We find a sextic equation which is satisfied by the normal mode frequencies, that is

$$c_0 + c_1 s + c_2 s^2 + c_3 s^3 + c_4 s^4 + c_5 s^5 + c_6 s^6 = 0 \quad (19)$$

where the polynomial coefficients are:

$$\begin{aligned}
c_6 &= (\tau_1\tau_2)^2 \\
c_5 &= 2\tau_1\tau_2(\tau_1+\tau_2) \\
c_4 &= (\tau_1\tau_2)[4 + (\tau_2/\tau_1)\sec^2\Psi_1 + (\tau_1/\tau_2)\sec^2\Psi_2 + (\tau_1\tau_2)\Delta] \\
c_3 &= 2[(\tau_1+\tau_2)\tau_1\tau_2\Delta + \tau_2\sec^2\Psi_1 + \tau_1\sec^2\Psi_2] \\
c_2 &= \sec^2\Psi_1\sec^2\Psi_2 + \tau_2^2\sec^2\Psi_1(\Delta_b - A(1/2)Y_{b1}\sin 2\Psi_1) + \\
&\quad \tau_1^2\sec^2\Psi_2(\Delta_b - BY_{b2}\sin 2\Psi_2) + 4\tau_1\tau_2\Delta_b \\
c_1 &= 2[\tau_2\sec^2\Psi_1(\Delta_b - A(1/2)Y_{b1}\sin 2\Psi_1) + \\
&\quad \tau_1\sec^2\Psi_2(\Delta_b - BY_{b2}\sin 2\Psi_2)] \\
c_0 &= \sec^2\Psi_1\sec^2\Psi_2[\Delta - (1/2)AY_{b1}\sin 2\Psi_1 - BY_{b2}\sin 2\Psi_2]
\end{aligned}$$

What is curious, and surprising, is that $\Delta > 0$ is not a necessary condition for the coefficients to be greater than zero.

It is worth discussing c_0 in some detail. The coefficient can become zero in a variety of ways. It is tempting to write

$$\begin{aligned}
c_0/2 &= \sec^2\Psi_2 A(\sec^2\Psi_1\cos\Phi_{b1} - Y_{b1}\tan\Psi_1) \\
&- \sec^2\Psi_1 2B(\sec^2\Psi_2\cos\Phi_{b2} + Y_{b2}\tan\Psi_2) \quad (20)
\end{aligned}$$

which suggests that $c_0 > 0$ when simultaneously,

$$Y_{b1} \leq \frac{2\cos\Phi_{b1}}{\sin 2\Psi_1} \quad \text{and} \quad Y_{b2} \geq \frac{2\cos 2\Phi_{b2}}{-\sin 2\Psi_2} \ . \quad (21)$$

However, in the bunch lengthening mode $\Delta \approx 0$ and so $A\cos\Phi_{b1} \approx 2B\cos 2\Phi_{b2}$. Consequently, $c_0 > 0$ if

$$-Y_{b2}\sin 2\Psi_2\cos\Phi_{b1} \geq Y_{b1}\sin 2\Psi_1\cos\Phi_{b2} \quad (22)$$

which is a constraint on the relative magnitudes of Y_{b1} and Y_{b2} . The equality will apply if $Y_{b1} \approx Y_{b2}$ and the cavities are detuned to present pure resistive loads and the synchronous phase angle is small; and this is roughly the case in the CERN PS Booster.

An alternative form for c_1 is

$$2[\Delta(\tau_2\sec^2\Psi_1 + \tau_1\sec^2\Psi_2) - (A\tau_2 Y_{b1}\tan\Psi_1 + 2B\tau_1 Y_{b2}\tan\Psi)] \ . \quad (23)$$

An alternative form for c_2 is

$$\begin{aligned}
&+ \sec^2\Psi_1\sec^2\Psi_2 - [\tau_2^2 AY_{b1}\tan\Psi_1 + \tau_1^2 2BY_{b2}\tan\Psi_2] \\
&+ \Delta\tau_1\tau_2[4 + (\tau_2/\tau_1)\sec^2\Psi_1 + (\tau_1/\tau_2)\sec^2\Psi_2] \ . \quad (24)
\end{aligned}$$

Given that $\Delta \approx 0$ in the bunch lengthening mode, the conditions $c_2 > 0$ and $c_1 > 0$ imply further constraints on the relative magnitude of Y_{b1} and Y_{b2} and suggest $\tau_1 > \tau_2$ is a desirable condition for stability.

2.2 Routh determinants

RH[1], RH[2], and RH[3] > 0 are all satisfied automatically provided $\tau_1, \tau_2 > 0$. RH[7]> 0 is identical with $c_0 > 0$.

RH[4]: $(\tau_2^2\sec^2\Psi_1 - \tau_1^2\sec^2\Psi_2)^2 + \tau_1\tau_2(\tau_1+\tau_2)\times$

$$[4(\tau_2\sec^2\Psi_1 + \tau_1\sec^2\Psi_2) + (A\tau_2^3 Y_{b1}\tan\Psi_1 + 2B\tau_1^3 Y_{b2}\tan\Psi_2)] > 0 \ .$$

This expression is exact and limits the amount of negative detuning. However, we may expect RH[4] to be satisfied in most practical cases, as usually $\Omega_s^2\tau_1^2 \ll 1$ and $\Omega_s^2\tau_2^2 \ll 1$ and typically $Y_{b1} \le |\tan\Psi_1|$ and $Y_{b2} \le |\tan\Psi_2|$.

RH[5]: $256(\tau_1\tau_2)^2(\tau_1+\tau_2)\times[\ldots] > 0$ is very long, but under the conditions $\Phi_{b2} \to 0$, $\Phi_{b1} \to 0$, the term $[\ldots]$ simplifies to

$$
\begin{aligned}
&16(\tau_1\tau_2)^2(\tau_1+\tau_2)[A\tau_2 Y_{b1}\tan\Psi_1 + 2B\tau_1 Y_{b2}\tan\Psi_2]+\\
&\sec^2\Psi_1\sec^2\Psi_2[4\tau_1\tau_2(\tau_1+\tau_2)(\tau_2\sec^2\Psi_1+\tau_1\sec^2\Psi_2)+\\
&+2(\tau_1\tau_2)^2(A\tau_2^2 Y_{b1}\tan\Psi_1 + 2B\tau_1^2 Y_{b2}\tan\Psi_2)]+\\
&4\Delta(\tau_1\tau_2)^2(\tau_1+\tau_2)[A\tau_2^3 Y_{b1}\tan\Psi_1 + 2B\tau_1^3 Y_{b2}\tan\Psi_2]+\\
&-\tau_1\tau_2(A\tau_2^3 Y_{b1}\tan\Psi_1 + 2B\tau_1^3 Y_{b2}\tan\Psi_2)^2+\\
&\tau_1\tau_2[A\tau_2^4 Y_{b1}\tan\Psi_1(4+\Delta\tau_2^2)\sec^2\Psi_1+\\
&+2B\tau_1^4 Y_{b2}\tan\Psi_2(4+\Delta\tau_1^2)\sec^2\Psi_2]+\\
&(\tau_1\tau_2)^2 A Y_{b1}\tan\Psi_1[(4\tau_1^2 - 8\tau_2(\tau_1+\tau_2)+\Delta\tau_1^2\tau_2^2)\sec^2\Psi_2\\
&-2\tau_1^2\sec^4\Psi_2] + (\tau_2^2\sec^2\Psi_1 - \tau_1^2\sec^2\Psi_2)^2+\\
&(\tau_1\tau_2)^2 2B Y_{b2}\tan\Psi_2[(4\tau_2^2 - 8\tau_1(\tau_1+\tau_2)+\Delta\tau_1^2\tau_2^2)\sec^2\Psi_1\\
&-2\tau_2^2\sec^4\Psi_1] . \qquad (25)
\end{aligned}
$$

Provided that $\Omega_s^2\tau_1\tau_2 \ll 1$ and $\Phi_{b1} = 0$ RH[5] is positive on the matched generator curve. For exact results, we need a relation between Y_{b2} and Y_{b1} for given R_1 & R_2, plus generator conditions.

RH[6]: $\tau_1^3\tau_2^3(\tau_1+\tau_2)[4\tau_1\tau_2(\tau_1+\tau_2) + \tau_2^3\sec^2\Psi_1 + \tau_1^3\sec^2\Psi_2]\times[\ldots.]$. The term $[\ldots]$ is too long to record here.

2.3 Approximations

Let us truncate the coefficients to dominant terms and then evaluate approximations for Routh determinants RH[5] and RH[6]. Let us assume $\Omega_s^2\tau_1\tau_2 \ll 1$ and $\Delta \to 0$ and drop these terms from the polynomial coefficients. c_6, c_5, c_1 and c_0 are unchanged.

$$
\begin{aligned}
c_4 &\approx \tau_1\tau_2[4 + (\tau_2/\tau_1)\sec^2\Psi_1 + (\tau_1/\tau_2)\sec^2\Psi_2] \qquad (26)\\
c_3 &\approx 2(\tau_2\sec^2\Psi_1 + \tau_1\sec^2\Psi_2) \qquad (27)\\
c_2 &\approx \sec^2\Psi_1\sec^2\Psi_2 . \qquad (28)
\end{aligned}
$$

RH[5]: After setting $\Phi_{b1} = 0$, $\Phi_{b2} = 0$ one finds RH[5]> 0 for most values of Y_{b1}, Y_{b2} and tuning angles, and in particular for the limit of large beam load and large detuning $Y_{b1} \approx |\tan\Psi_1|$ and $Y_{b2} \approx |\tan\Psi_2|$. Further, RH[5]$> 0$ below and on the matched generator curves $Y_{b1} = \tan\Psi_1$, $Y_{b2} = -\tan\Psi_2$ when $\Omega_s^2\tau_1^2 \ll 1$ and $\Omega_s^2\tau_2^2 \ll 1$. Lastly, $c_1 > 0$ and/or $c_0 > 0$ are not necessary conditions for RH[5]> 0.

RH[6] is a very complicated expression. However, in the limit that c_0 and c_1 are small (i.e. the bunch lengthening mode), so that powers and products are negligible, then RH[6] is equal to

$$
\begin{aligned}
&16[c_1\sec^2\Psi_1\sec^2\Psi_2 - 2c_0(\tau_2\sec^2\Psi_1 + \tau_1\sec^2\Psi_2)]\times\\
&[4\tau_1\tau_2(\tau_1+\tau_2)(\tau_2\sec^2\Psi_1+\tau_1\sec^2\Psi_2)+(\tau_1^2\sec^2\Psi_2-\tau_2^2\sec^2\Psi_1)^2]\\
&+8c_0c_1\tau_1^2\tau_2^2[8(\tau_1+\tau_2)^2+2[(\tau_2^3/\tau_1)\sec^2\Psi_1+(\tau_1^3/\tau_2)\sec^2\Psi_2]\times\\
&\tau_2(\tau_2-\tau_1)\sec^2\Psi_1 + \tau_1(\tau_1-\tau_2)\sec^2\Psi_2] . \qquad (29)
\end{aligned}
$$

N.B. We made no assumptions about Φ_{b1} or Φ_{b2} Clearly, $c_1 > 0$ or $c_0 < 0$ is a necessary condition for RH[6]> 0. If $c_0 \times c_1 \to 0$, then the criterion becomes

$$c_1\sec^2\Psi_1\sec^2\Psi_2 \;>\; 2c_0(\tau_2\sec^2\Psi_1 + \tau_1\sec^2\Psi_2) \qquad (30)$$

and so $c_0\sqrt{\tau_1\tau_2} < c_1/2$ assuming $\tau_1 \approx \tau_2$, or explicitly

$$A\tau_1 Y_{b1}\cos^2\Psi_1\sin 2\Psi_1 + 2B\tau_2 Y_{b2}\cos^2\Psi_2\sin 2\Psi_2 > 0 . \qquad (31)$$

If there is no constraint on moduli then $c_1 \neq 0$ or $c_0 \neq 0$ is a necessary condition for RH[6]$\neq 0$.

3 WITH PHASE LOOP

We supplement previous working with the beam-phase loop equation: $s\phi_{g1,s} = K_p(\phi_{b1,s} - \phi_{v1,s}) = K_p\phi_{b1,v1}$. In the case of a *single* RF system, this choice of the sign of K_p would lead to the conclusion $K_p > 0$. In this present model, there is no control over the phase of the higher harmonic, and $\phi_{g2,s} \equiv 0$. Due to the presence of the loop, the general steady state generator conditions have been substituted in the determinant.

This leads to a heptic equation

$$d_0 + d_1 s + d_2 s^2 + d_3 s^3 + d_4 s^4 + d_5 s^5 + d_6 s^6 + d_7 s^7 = 0 \quad (32)$$

where the polynomial coefficients are: $d_7 = c_6$, $d_6 = c_5$,

$$
\begin{aligned}
d_5 &= c_4 + K_p\tau_1\tau_2^2 \qquad (33)\\
d_4 &= c_3 + K_p\tau_2[2\tau_1 + \tau_2(\sec^2\Psi_1 - Y_{b1}\tan\Psi_1)] \qquad (34)\\
d_3 &= c_2 + K_p[\tau_1\sec^2\Psi_2 - 2B\tau_1\tau_2^2\\
&\quad +2\tau_2(\sec^2\Psi_1 - Y_{b1}\tan\Psi_1)] \qquad (35)\\
d_2 &= c_1 + K_p[(\sec^2\Psi_1 - Y_{b1}\tan\Psi_1)\times\\
&\quad (\sec^2\Psi_2 - 2B\tau_2^2) - 4B\tau_1\tau_2] \qquad (36)\\
d_1 &= c_0 + 2BK_p[2\tau_2(Y_{b1}\tan\Psi_1 - \sec^2\Psi_1)\\
&\quad -\tau_1(Y_{b2}\tan\Psi_2 + \sec^2\Psi_2)] \qquad (37)\\
d_0 &= 2BK_p(-\sec^2\Psi_1 + Y_{b1}\tan\Psi_1)(\sec^2\Psi_2 + Y_{b2}\tan\Psi_2) \qquad (38)
\end{aligned}
$$

Coefficient d_0 has *important* implications. Either

- $K_p < 0$ and $[\sec^2\Psi_1 + \ldots] > 0$ and $(\cos\Phi_{b2}\ldots) > 0$
- $K_p > 0$ and $Y_{b1} > \sec\Psi_1/\sin(\Psi_1 - \Phi_{b1})$ & $(\cos\Phi_{b2}\ldots) > 0$, or $[\sec^2\Psi_1 + \ldots] > 0$ & $Y_{b2} < -2\cos\Phi_{b2}/\sin 2\Psi_2$.

On the matched generator curves c_0 is equal

$$-BK_p[1 + \tan\Phi_{b1}\tan\Psi_1](\cos^2\Phi_{b2} - \sin^2\Phi_{b2}\tan^2\Psi_2)\sec\Phi_{b2} \qquad (39)$$

which is negative unless $K_p \times (-\tan^2\Psi_2 + 1/\tan^2\Phi_{b2}) < 0$.

3.1 Routh determinants

RH[1] and RH[2] > 0 are satisfied automatically. RH[8] is identical with $d_0 > 0$. RH[3]:

$$
\begin{aligned}
&[8\tau_1\tau_2(\tau_1+\tau_2) + 2\tau_2^3\sec^2\Psi_1 + 2\tau_1^3\sec^2\Psi_2+\\
&+K_p\tau_1\tau_2^3(1 + Y_{b1}\tan\Psi_1 - \tan^2\Psi_1)] \;>\; 0 \qquad (40)
\end{aligned}
$$

is always satisfied provided $\Psi_1 \ge 0$ and

$$K_p\tau_1 < 2[1 + (\tau_1/\tau_2)^3\tan^2\Psi_2/\tan^2\Psi_1] .$$

RH[4]: $0 \le K_p\tau_1 < 1$ and $0 \le K_p\tau_2 < 1$ will be a sufficient condition for stability in most cases. Contrarily in the limit of large gain one finds:

$$\text{RH[4]} \propto -K_p^2\tau_1^2\tau_2^5(\sec^2\Psi_1 - Y_{b1}\tan\Psi_1)^2 \;>\; 0 ; \qquad (41)$$

this condition cannot be met, and so K_p must be limited. RH[5,6,7] are too lengthy to report or comment upon.

4 REFERENCES

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