NEW MATHEMATICAL MODEL OF AN INFINITE CAVITY CHAIN

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1 INTRODUCTION

The chain of coupled cavities are widely used in the RF-engineering. Slow wave structures on their base are most common in the accelerator technology, finding various applications as well in RF-devices designed to generate and amplify electromagnetic waves (see, for example, [1,3]). Since resonant properties of each cavity can be described by equations that resemble in their outward appearance the classic equations of the resonant circuits, then, a coupled cavity chain should be described similarly. Such an approach to the study of properties of coupled cavity chain (a method of equivalent circuits) is very useful to model rf cavities. Its advantage over purely electrodynamic methods is in its explicitness and a relative simplicity of the mathematical analysis which is of supreme importance for the stage of primary electrodynamic properties study and conceptual design of the system. It is especially manifest in the development of complex structures: a chain of cavities coupled through slots [4], a biperodic or compensated structure [5,6], a detuned structure [7] and others. However, justification of such models must be made on the base of the electrodynamic approach which simultaneously gives their accuracy. The main question in utilization of the circuit model for description of a coupled cavity chain is the possibility of truncation the number of circuits under consideration and their connections, since a precise account of all these factors would do more than simply eliminate the advantages of this approach: it would make this problem mathematically unresolvable. By doing such truncation we have to take into account the following circumstance. Since commonly the analysis of characteristics has to be made within a confined frequency range, then the total initial set of circuits can be broken down into two classes: the resonant one, representing modes with eigen-frequencies which values are close to the frequency range under the consideration, and the non-resonant one. If the couplings of the resonant circuits form the main frequency properties of the system, then the presence of the non-resonant circuits form the properties of such couplings. Although the amplitude of each non-resonant mode is small , their total effect on resonant circuit couplings is considerable. Based on the rigorous electrodynamic approach, in two cavity coupling problem we could separate the above circuit types and also bring down the study of influence of the non-resonant modes to the field coupling on the boundaries dividing the cavities [9,11]. This approach allowed to preserve the explicitness of the model as a system of coupled resonant circuits and calculate accurately the necessary coupling coefficients. In this work this method is used to describe an infinitely long chain of cylindrical cavities coupled through central holes in sidewalls. The focus of attention is paid to the calculation of the value of coupling of cavities which have no immediate contact. The number of couplings to be taken into account determines not only the slow-wave structure properties, but also a possibility of their matching and tuning. The latter is of added importance for development of inhomogeneous structures. As distinct from the previous papers [3,5,6,12], we have managed to elaborate on a model, allowing consecutively to take into consideration any number of couplings.

2 BASIC EQUATIONS

Let us consider an infinite chain of similar ideally conducting co-axial cylindrical cavities (disk-loaded wavequide) coupled through cylindrical holes with the radius *a* in the dividing walls of the thickness *t*. The radii and lengths of the cavities we denote by *b* and *d*. The disks of the *i*-th cavity we denote by the indexes *i* and (*i*-1). In order to construct a mathematical system model under consideration we will use a method of partial cross-over regions [13]. Using the method, similar to the one in [1-3], one can show that the set of equations, describing the system under consideration, has the form:

$$(\omega_{0,1}^2-\omega^2)a_{0,1}^{(i)}=-\omega_{0,1}^2\frac{2}{3\pi}\frac{a^3}{J_1^2(\lambda_1)b^2d}\times$$

$$\left[2a_{0,1}^{(i)}\Lambda_{-,1}+\sum_{k=1}^{\infty}(a_{0,1}^{(i+k)}+a_{0,1}^{(i-k)})(\Lambda_{-,k+1}-\Lambda_{+,k})\right],\qquad(1)$$

where $a_{0,1}^{(i)}$ - the amplitude of $E_{0,1,0}$ -mode in the *i*-th cavity,

$$\Lambda_{\pm,k}=J_0^2(\lambda_1 a/b)\sum_{s'}w_{\pm,s'}^{(k)}/[\lambda_{s'}^2-(\lambda_1 a/b)^2],\qquad(2)$$

and $w_{\pm,s}^{(k)}$ are the solutions of the following sets of linear algebraic equations:

$$w_{\pm,n}^{(1)}+\sum_s\left[w_{\pm,s}^{(1)}G_{n,s}^{(1,1)}+w_{\mu,s}^{(1)}G_{n,s}^{(1,2)}+w_{-,s}^{(1)}G_{n,s}^{\left(1,{2 \atop 1}\right)}\right]=$$

$$=\sum_s w_{-,s}^{(2)}G_{n,s}^{(2,{1 \atop 2})}+3\pi f_n^{(1,{2 \atop 1})}/\left[\lambda_n^2-\left(\lambda_1 a/b\right)^2\right],\quad(3.1)$$

$$w_{\pm,n}^{(k)}+\sum_s\left[w_{\pm,s}^{(k)}G_{n,s}^{(1,1)}+w_{\mu,s}^{(k)}G_{n,s}^{(1,2)}\right]=$$

$$=\sum_s\left[w_{\mu,s}^{(k\pm1)}G_{n,s}^{(2,1)}+w_{\pm,s}^{(k\mu1)}G_{n,s}^{(2,2)}\right]. \qquad (3.2)$$

In Eqs.(3.2) $k=2,3...\infty$. The closed set of equations (1-3) describes rigorously the electrodynamic system under consideration. Eqs.(1) describe the coupling of the infinite chain of the resonant circuits, with the coupling coefficients $\Lambda_{\pm,k}$ being frequency functions.

From Eq.(1) one can deduce that the electric field tangential components in the circular regions, through which i-th cavity is connected with others elements of the system under consideration , are only determined via the fundamental mode $E_{0,1,0}$ amplitudes of all cavities. The coefficients $w_{\pm,s}^{(i)}$ are proportional to the expansion coefficients of tangential electric field with the complete set of functions $\left\{J_1(\lambda_s r/a)\right\}$ on the right and the left hole cross-sections of the i-th disk.

Thus, the problem of coupled cavities has been rigorously reduced to the problem of the coupling of electric fields (see, Eqs.(3)), which are determined in circular regions $r \le a$.

3 RESULTS OF THE ANALYSIS AND NUMERICAL SIMULATION

The coupling of fields on various disks are described in Eqs.(3) by the terms which contain factors $G_{n,s}^{(2,i)}$. This is confirmed by the fact that at $a\to 0$ and $t=0$ $G_{n,s}^{(2,i)}\to 0$, while $G_{n,s}^{(1,i)}$ tend to constant values, independent of a. In this case Eqs.(3) have the following solution

$$w_{+,n}^{(1)}=w_{-,n}^{(1)}=6(\sin(\lambda_n)-\lambda_n\cos(\lambda_n))/\lambda_n^2,$$

$$w_{+,n}^{(k)}=w_{-,n}^{(k)}=0,\quad k=2,3...\infty,$$

at which $\Lambda_{-,1}=\Lambda_{+,1}=1$ and all the remaining values of the normalized coupling coefficients are equal to zero. At these conditions the set (1) coincides with the equations describing an infinite cavity chain obtained in the quasi-static approximations [14-16]. Eqs.(1) have the solution of the kind $a_{0,1}^{(n)}=a_0\exp(i\,n\phi)$, where a_0 is the constant, while ϕ is determined from the following equation:

$$(\omega_{0,1}^2-\omega^2)=-\omega_{0,1}^2\frac{4}{3\pi}\frac{a^3}{J_1^2(\lambda_1)b^2 d}\times$$

$$\times\left[\rho_0(\omega)+\sum_{k=1}^{\infty}\rho_k(\omega)\cos(k\phi)\right], \qquad (4)$$

where

$$\rho_0(\omega)=\Lambda_{-,1}(\omega),\ \rho_k(\omega)=\left(\Lambda_{-,k+1}(\omega)-\Lambda_{+,k}(\omega)\right).$$

From Eq.(4) it follows that in the general case in order to determine the phase shift between cavities it is necessary that couplings of all disks be taken into account. However, as numerical simulations indicate, the contribution of "long range" couplings is small and one can confine oneself to considering field couplings on the finite number of disks. There, since we had used some symmetry relationships, it is necessary to observe the strict correlation between the number of terms in the sum over k in Eq.(4) and the number of equations taken into account in Eqs.(3). Results of numerical analysis of Eq.(4}) are presented below. Tab.1 gives the calculated values[1] of phase shift (in degrees) per cell for the cavity chains with such geometrical dimensions that ensure phase shifts to occur close to $\phi=2\pi/3$, $\pi/2$, and $\pi/3$. The operation frequency is f_0=2797.0 MHz (λ_0= 10.7183 cm). We have: the results in column (1) correspond to the case of non-coupling disks, (2) - two disks, (4) - four, (6) - six, (8) - eight disks are coupled.

From Tab.(1) it follows that the influence of coupling of different disk fields on the buildup of a certain phase shift depends both on the spacing of the disks and on the hole dimensions.

Table 1.

Calculated values of the phase shift (in degrees) per one cavity for various cavity chains

a/λ_0	(1)	(2)	(4)	(6)	(8)
D = λ_0/3					
0.08	120.163	120.025	120.012	120.012	120.012
0.14	120.583	120.071	120.012	120.013	120.013
D = λ_0/4					
0.08	88.698	90.183	90.010	90.012	90.012
0.11	88.427	90.458	89.978	89.986	89.986
0.14	88.393	90.919	89.993	90.017	90.018
D = λ_0/6					
0.08	54.873	61.994	60.059	60.065	60.065
0.11	55.855	63.810	60.084	60.078	60.083
0.14	57.458	65.776	60.161	60.081	60.100

Thus, for instance, in the case of disk-loaded structures operating in the $\phi=\pi/3$ mode, even at small values of the hole radius, it is necessary to take into account field coupling of four disks, while at large one - six disks. In the case of disk-loaded structures operating in the $\phi=\pi/2$ mode it is necessary to take into account field coupling of four disks. In the case of the most commonly used disk-loaded structures operating in the $\phi=2\pi/3$ mode only coupling of fields of two disks should be taken into account for a broad range of hole

[1] Our results are in good agreement both with the experimental data, given in [3], and with the calculation results performed within the program developed on the base of partial region technique [17].

radii. The dependence of corresponding coefficients on frequency, in general, seriously influences on the electrodynamic characteristics of the system under consideration. Tab.2 presents calculation results of the relationship of phase shift per cavity versus frequency (dispersion relation) for a homogeneous disk-loaded waveguide with $D = \lambda_0 / 3$ and $a / \lambda_0 = 0.14$.

Table 2

Phase shift (in degrees) vs frequency

f, GHz	A	B	C
2.727	25.15	25.23	27.59
2.737	45.12	45.24	48.19
2.747	59.63	59.80	63.50
2.757	72.20	72.48	77.01
2.767	84.01	84.35	89.84
2.777	95.57	96.03	102.72
2.787	107.39	108.01	116.35
2.797	120.07	120.93	131.94
2.807	134.73	136.02	153.34
2.877	155.07	157.81	-

The column (A) corresponds to the case $\rho_i = \rho_i(\omega)$, column (B) - $\rho_i = \rho_i(\omega_{010})$, column (C)- $\rho_i = \rho_i(0)$ (quasistatic case). From the calculations it follows that ρ_i vs. ω relationship, even within the passband, exercises an influence upon phase shift.

From the results above, one can deduce that cavity chains of any geometry (homogeneous and inhomogeneous) with $D \geq \lambda_0/3$ and $a/\lambda_0 \leq 0.14$ can be described very accurately by the coupled circuit model, wherein each resonant circuit is coupled to two neighboring ones:

$$(\omega_{0,1}^2 - \omega^2)\, a_{0,1}^{(i)} = -\omega_{0,1}^{(i)\,2} \times$$
$$\times\left[a_{0,1}^{(i)}\,\Gamma^{(i)}(\omega) - \left(a_{0,1}^{(i+1)}\,\Gamma_+^{(i)}(\omega) + a_{0,1}^{(i-1)}\,\Gamma_-^{(i)}(\omega)\right)\right], \quad (5)$$

where the coefficients $\Gamma^{(i)}$, $\Gamma_\pm^{(i)}$ for *i*-th cavity will be determined by two values of the radii of coupling holes, through which this cavity is connected with adjacent ones, geometrical dimensions of the (*i*-1, *i*, *i*+1)-th cavities and frequency. The results of studies of inhomogeneous cavity chains on the base of Eq.(5) will be presented in a future paper.

4 CONCLUSION

In this paper on the base of a rigorous electrodynamic approach we have developed a mathematical model of a cylindrical cavity chain with electric coupling. This model combines the model of the equivalent coupled circuit chain and an accurate description of the non-resonant field influence. The above approach can be also used in the case of magnetic coupling. In this case the problem of accurate description of the potential fields on the holes and slots (see, for example, [18]) will be easier, because within the frame of the partial cross-over regions method the subset of irrotational modes is a part of the complete set of modes that one has to use to expand fields with. This technique is easily transformed for the case of inhomogeneous structures. Then, there is a possibility to control rigorously the effects of "long-range" coupling of cavities.

To this day, the equivalent circuit model was an only approximate one at large couplings. In this case, one had to determine the circuit chain parameters from the measured dispersion curves of the passbands. The above method imbues one with hope that this model can give sufficiently accurate description of the characteristics of the coupled cavity chain at large couplings.

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