Second Order Error Calculation Package for Beam Transfer Lines

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Abstract

In the aim of investigating the tolerances of final focus systems for linear colliders, a program FFSER2 has been written to calculate up to the second order the effects of misalignments and field errors on the transfer map of beam transfer lines. For its specific application to final focus systems, it also calculates the corresponding tolerances based on the spot size growth and luminosity loss at the interaction point. The program is interfaced with the graphics package PAW, allowing the analytic display of the calculated errors and tolerances.

1 INTRODUCTION

Effects on beam due to magnet misalignments or field errors can drastically damage the collider performance in terms of luminosity. Instead of calling upon a multitude of trackings in order to quantify these effects, a semi-analytic approach had been developed permitting to deal with this problem in a systematic way; it divides into two stages:

- the first one (Section 3), applicable to the study of any beam transfer line, consists in estimating the transfer map of the line considered up to a given order in the errors and in the non-linearities of the Hamiltonian.
- the second one (Section 4), specific to final focus system studies, consists in computing the spot-size or the luminosity at the interaction point as an analytic function of the transfer map (including the errors).

2 EQUATION OF MOTION

2.1 Recalls and notations

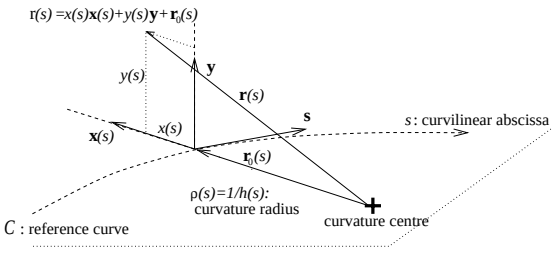

Figure 1: Reference curve

We consider a plane reference curve $\mathcal{C}$ of curvature $h(s)$, made of a succession of arcs (at the dipole location) and straight segments (at the drift, quadrupole and sextupole location). The position of any particle in the three-dimensional Euclidean space is then defined with respect to the curve $\mathcal{C}$ by $\mathbf{r}(s) = x(s)\mathbf{x}(s) + y(s)\mathbf{y} + \mathbf{r}_0(s)$. We also introduce a reference particle $\mathcal{P}_0$ of charge q, of velocity $v_0 = \beta_0 c$ and momentum $P_0 = m\gamma_0 v_0$ which corresponds to the design energy $E_0 = \gamma_0 mc^2$; it is well-known that the $\mathcal{P}_0$ trajectory and the curve $\mathcal{C}$ coincide if the vectors $\mathbf{r}$ and $\mathbf{r}_0$ are the same at the line entrance and if the magnetic field is vertical (i.e parallel to $\mathbf{y}$) on the curve $\mathcal{C}$ and equal to hP_0/q. In practice, these conditions are never fulfilled (due to magnet misalignments or field errors) and the position of any particle in the phase space can be parameterized by the six-dimensional vector X defined hereafter:

$$X \stackrel{\text{def}}{=} \begin{cases} x & \text{: horizontal coordinate} \\ p_x = P_x/P_0 & \text{: normalized horizontal momentum} \\ y & \text{: vertical coordinate} \\ p_y = P_y/P_0 & \text{: normalized vertical momentum} \\ Dt = -c(t - s/v_0) & \text{: delay with respect to the ref. particle} \\ \delta = (E - E_0)/(P_0 c) & \text{: normalized difference of energy} \end{cases} \quad (1)$$

which are conjugated by the following Hamiltonian [1, 2] (s is the independent variable):

$$\mathcal{H} = -(1 + hx)\left[\sqrt{1 + 2\frac{\delta}{\beta_0} + \delta^2 - p_x^2 - p_y^2} + K_s\right] + \frac{\delta}{\beta_0} \quad (2)$$

where $K_s \equiv K_s(x, y, s)$ is linked to the longitudinal vector potential A_s by the relation $K_s = qA_s/P_0$[1].

2.2 Map and equation of motion

The equation of motion is then written in a vectorial way:

$$\frac{dX}{ds} = \mathbf{F}(X, s, \lambda) \quad \text{with}$$
$$\mathbf{F} \stackrel{\text{def}}{=} (\partial_{X_2}\mathcal{H}, -\partial_{X_1}\mathcal{H}, \partial_{X_4}\mathcal{H}, -\partial_{X_3}\mathcal{H}, \partial_{X_6}\mathcal{H}, -\partial_{X_5}\mathcal{H}) \ , \quad (3)$$

λ being a parameter set containing the multipolar strengths occurring in K_s. $\mathbf{F}$ being not linear in X, the solutions of Eq. 3 do not depend linearly on the initial conditions X_0:

$$X(s) \stackrel{\text{def}}{=} \mathcal{M}(X_0, s, \lambda) = dX(s, \lambda) + R(s, \lambda)X_0 + T(s, \lambda)[X_0, X_0] + U(s, \lambda)[X_0, X_0, X_0] + \ldots \quad (4)$$

(i.e. $X_i = dX_i + R_{ij}X_{0_j} + T_{ijk}X_{0_j}X_{0_k} + \ldots , 1 \le i \le 6$) where dX represents the beam orbit deviation (occurring in the case where, due to errors of the line, the reference particle does not follow the reference curve), and where R, T and U are the first, second and third order transport matrices respectively. In practice, the computation of these quantities is done separately for each magnet of the line; their values at the line exit is then obtained by successive concatenations. More precisely, if $\mathcal{M}_{1,2}$ are two maps associated to two consecutive magnets, the problem is the computation of the algebraic product $\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1$. It can be shown that the exact computation of quantities $(dX,\ R,\ T,\ \ldots)$ (associated to $\mathcal{M}$) is impossible if the orbit deviation dX_1 (associated to $\mathcal{M}_1$) is non-zero [1]. The idea is then to consider a design optics (without errors) characterized by a parameter set λ_0 for which the reference particle trajectory and the curve $\mathcal{C}$ are the same (i.e.

[1]For the sake of simplification, the vector potential will be assumed purely longitudinal as is the case for multipoles without fringing fields.

$\mathbf{F}(X,s,\lambda_0) \equiv 0$ implying $dX_{1,2}(\lambda_0)=0$) and to develop the quantities $(dX_{1,2},\ R_{1,2},\ T_{1,2},\ \ldots)$ around this optics:

$$\begin{aligned} dX_{1,2}(\lambda) &= \delta\lambda \left(\partial_\lambda dX_{1,2}\right)_{\lambda=\lambda_0} + \ldots = dX_{1,2}^{(1)} + \ldots \\ R_{1,2}(\lambda) &= R_{1,2}^{(0)} + R_{1,2}^{(1)} + R_{1,2}^{(2)} + \ldots \text{ and so on.} \end{aligned} \tag{5}$$

The problem is now to fix a limit to the perturbative developments occurring in Eqs. 4 and 5. This limit will be discussed in Section 4 and is detailed hereafter. For a given magnet i of the line, all error sources $\epsilon_{i_\alpha} \equiv \delta\lambda_{i_\alpha}$ affecting its map up to the T-matrix will be considered and the following eight quantities will be computed:

- the orbit deviation and the R-matrix at the order 2 in the errors (i.e the quantities $dX^{(1,2)}$ and $R^{(0,1,2)}$).
- the T-matrix at the order 1 in the errors (i.e. $T^{(0,1)}$).
- the U-matrix at the order 0 in the errors (i.e. $U^{(0)}$).

This truncation choice is self-consistent [1] in the sense that all the quantities previously mentioned can then be derived at the line exit by successive concatenations.

3 MAP COMPUTATION

In order to derive the eight quantities previously cited, two different approaches will be adopted depending on the type of error considered: field errors or misalignments.

3.1 Field errors

To deal with field errors, a differential-type method is used. Considering the equation of motion (3), the vectorial function $\mathbf{F}(X,s,\lambda)$ is developed in a double perturbative series in X and in the errors $(\epsilon_{i_\alpha}) = \lambda - \lambda_0$:[2]

$$\begin{aligned} \frac{dX}{ds} = \mathbf{F}(X,s,\lambda) &= \sum_{k,l} \epsilon^l \left(\partial_{X^k}\partial_{\lambda^l}\mathbf{F}\right)_{(X=0,\,\lambda=\lambda_0)} [\overbrace{X,\ldots,X}^{\times k}] \\ &\stackrel{\text{def}}{=} F_0^{(1)} + F_0^{(2)} + \ldots + \left(F_1^{(0)} + F_1^{(1)} + \ldots\right)X + \\ &\quad \left(F_2^{(0)} + F_2^{(1)} + \ldots\right)[X,X] + \ldots \end{aligned} \tag{6}$$

By writing the vector X in function of the initial conditions X_0 (Eq. 4), by using the perturbative development of the quantities $(dX,\ R,\ T,\ \ldots)$ (Eq. 5) and finally, by identifying on either side of the previous equation the terms of same homogeneity in $(X_0\,,\epsilon)$, the differential equations verified by the eight quantities $dX^{(1,2)}$, $R^{(0,1,2)}$, $T^{(0,1)}$ and $U^{(0)}$ are easily obtained:

- the matrix $R^{(0)}$ (usual R-matrix associated to the magnet without field errors) verifies

$$dR^{(0)}/ds = F_1^{(0)}(s)\, R^{(0)}(s) \ \text{ and } \ R^{(0)}(0) = \mathbf{1}\ . \tag{7}$$

- the seven quantities $(dX^{(1,2)},\ R^{(1,2)},\ T^{(0,1)},\ U^{(0)})$ noted $(\mathcal{Q}_i,\ 1\le i\le 7)$ verify a differential equation of type

$$d\mathcal{Q}_i/ds = F_1^{(0)}(s)\, \mathcal{Q}_i(s) + G_i(s) \text{ and } \mathcal{Q}_i(0) = \mathbf{0}\,, \tag{8}$$

where G_i is an object of same type as $\mathcal{Q}_i$ (vector, matrix or tensors with 3 or 4 indices) which depends on the quantities $F_k^{(l)}$ (which are known since the Hamiltonian is known) and $(\mathcal{Q}_j\,,\ \mathbf{j}<\mathbf{i})$. The solution of Eq. 8 is then given by

$$\mathcal{Q}_i(s) = R^{(0)}(s) \int_0^s ds'\ \left(R^{(0)}(s')\right)^{-1}\, G_i(s') \tag{9}$$

so that, if the expression of the matrix $R^{(0)}$ is known (which is the case for multipoles without fringing fields), the quantities $(\mathcal{Q}_i,\ 1\le i\le 7)$ can be successively computed. This computation method has been adopted for the study of the right dipole, quadrupole and sextupole (which are the classical magnets occurring in beam transfer lines), taking into consideration the following errors:

- 5 field errors for the right dipole (right dipolar field error, right and skew quadrupolar and sextupolar field errors).
- 3 field errors for the right quadrupole (right quadrupolar field error, right and skew sextupolar field errors).
- the right sextupolar field error for the right sextupole.

Finally, note that for a right $2m$-pole ($1\le m\le 3$), the skew $2m$-polar field error as well as the skew and right $2n$-polar ($n<m$) field errors have not been considered here since the latter can be also studied in terms of misalignments.

3.2 Misalignments

The method used to deal with magnet misalignments is purely algebraic. Six independent parameters are required to define the real position of a given magnet (considered as a rigid object) with respect to its design position: three translation parameters $(dx\,, dy, ds)$ along the horizontal, vertical and longitudinal axis respectively and three successive rotations (keeping invariant the middle point of the entrance face of the translated magnet) of respective angle θ (azimuth angle around **x**), ϕ (elevation angle around **y**) and ψ (roll angle around **s**). It can then be shown [1] that the map $\mathcal{M}(\epsilon)$ associated to the misaligned magnet is given by the composition of the three following transformations:

$$\mathcal{M}(\epsilon) = \mathcal{M}_{\text{out}}(\epsilon; L, h) \circ \mathcal{M}_0 \circ \mathcal{M}_{\text{in}}(\epsilon)\ , \tag{10}$$

where $\mathcal{M}_0 \stackrel{\text{def}}{=} \mathcal{M}(\epsilon=0)$ is the map of the magnet in its ideal position and where the effective maps $\mathcal{M}_{\text{in}}$ and $\mathcal{M}_{\text{out}}$ are canonical transformations associated to the magnet pole face rotations and to the changes of spatial coordinates at its exit and entrance. The latter depend only on the misalignments ($\mathcal{M}_{\text{out}}$ depends also on the geometrical characteristics of the magnet, i.e. its length L and curvature h) and not on the type of the magnet considered. They are analytically computable in terms of the algebraic quantities previously defined (i.e. $dX_{\text{in}}, dX_{\text{out}}, R_{\text{in}}, \ldots$).

4 APPLICATION TO FINAL FOCUS SYSTEMS (FFS)

To summarize, for any beam transfer line, these methods permit to calculate the following quantities:

- the matrices $R^{(0)}$, $T^{(0)}$ and $U^{(0)}$ of the ideal line.
- the first derivatives of the beam orbit deviation with respect to all the line errors (field errors and misalignments)

[2] For instance, for the study of a right dipole, the parameter set λ_0 is reduced to the nominal dipolar strength $K_0^+ \equiv h$ and the set (ϵ_α) contains the right or skew multipolar field errors of type $\delta K_m^{\pm},\ m\ge 0$.

as well as the ones of the matrices R and T, i.e. the quantities $dX^{(1)}$, $R^{(1)}$ and $T^{(1)}$ at the line exit.
• the second derivatives $dX^{(2)}$ and $R^{(2)}$ at the line exit.
Considering now the particular case of a final focus system, we are going to show that this method can be used for tolerance computations based on the spot-size growth or luminosity loss at the interaction point (IP).

4.1 Beam matrix and luminosity

In final focus systems, the main part of non-linearities is suppressed at the interaction point (for instance, the chromatic aberrations generated by the strong quadrupolar field of the last doublet is compensated upstream by sextupole pairs). The map $\mathcal{M}^*$ associated to such a beam line is therefore **globally** linear (of course, there exist local zones where the non-linearities are very high):

$$\mathcal{M}^*(X_0) \approx dX + RX_0 = dX + (R^{(0)} + \delta R)X_0\,. \qquad (11)$$

In this case, the beam matrices Σ_0 and Σ^* (i.e. the covariance matrices of the beam distribution at the line entrance and at the IP respectively) are linked by the usual relation,

$$\Sigma^* = R\Sigma_0 R^{\mathrm{T}} = \Sigma^{(0)^*} + \delta Q \Sigma^{(0)^*} + \Sigma^{(0)^*} \delta Q^{\mathrm{T}} + \delta Q \Sigma^{(0)^*} \delta Q^{\mathrm{T}}\,, \quad (12)$$

where $\Sigma^{(0)^*} \stackrel{\mathrm{def}}{=} R^{(0)}\Sigma_0 R^{(0)\,\mathrm{T}}$ is the IP beam matrix associated to the ideal line and $\delta Q \stackrel{\mathrm{def}}{=} \delta R R^{(0)^{-1}}$. The beam matrix fluctuations and therefore the spot-size growth at the IP contain then a linear term in the R-matrix defects. Hence, in order to quantify these fluctuations up to the second order in the errors, the derivation of quantities $R^{(1)}$ but also $R^{(2)}$ at the collision point becomes imperative, which justifies the truncation choice made in Section 3.
This argument holds also for the luminosity computation at the IP. Indeed, the geometric luminosity (i.e. calculated by neglecting the beam-beam interaction) possesses an analytic form of type $\mathcal{L}(dX, \delta R)$ (see Ref. [3]) which, developed at the second order, also presents a linear term in δR.
To summarize, by estimating the function $\mathcal{L}(dX, \delta R)$ by its Taylor series at the order two (see Ref. [1]) and by using the perturbative expansion of quantities dX and δR at the second order in the errors (Section 3), we finally obtain for the luminosity loss (or the spot-size growth) the following development in the errors of the line:

$$\frac{\delta\mathcal{L}}{\mathcal{L}_0} = \sum_{i_\alpha} a^{(1)}_{i_\alpha}\epsilon_{i_\alpha} + \sum_{i_\alpha, j_\beta} a^{(2)}_{i_\alpha, j_\beta}\epsilon_{i_\alpha}\epsilon_{j_\beta} + \circ(\epsilon^2)\,. \qquad (13)$$

4.2 Tolerance computation

It is now a matter of specifying the distribution of these errors along the line. In this paper, we have considered the following two situations.
• all the magnets except one, which is affected by an error ϵ_{i_α}, are perfectly aligned and tuned. By imposing a maximum luminosity loss of 2%, the tolerance on ϵ_{i_α} is then obtained by solving the second degree equation $\left|a^{(1)}_{i_\alpha}\epsilon_{i_\alpha} + a^{(2)}_{i_\alpha, i_\alpha}\epsilon^2_{i_\alpha}\right| \leq 2\%$. The result of such a computation is illustrated in Fig 2.1 giving the tolerances to field errors of TESLA FFS dipoles and quadrupoles [5]. Here the inverse tolerances are plotted so that the highest bars correspond to the tightest tolerances.

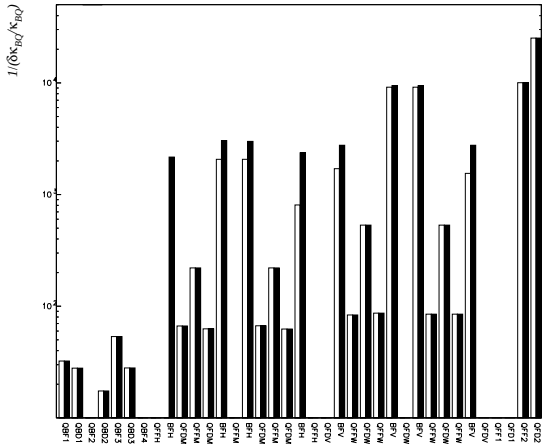

Figure 2: Tolerances to field errors of TESLA FFS dipoles and quadrupoles for 2% luminosity loss with (white bars) and without (black bars) vertical IP offset correction

• the errors ϵ_{i_α} are uncorrelated and of mean value equal to zero. Let us consider for instance the case where only the magnet transverse motions are taken into account. In this model and for the TESLA FFS magnets, Eq. 13 gives the following numerical result for small luminosity losses [5]:

$$\frac{\delta\mathcal{L}}{\mathcal{L}_0} = -2\% \times \left\{ \left(\frac{\sigma_x^{(mag)}}{558\,\mathrm{nm}}\right)^2 + \left(\frac{\sigma_y^{(mag)}}{203\,\mathrm{nm}}\right)^2 \right\}$$

(for only one FFS and for the case where the IP vertical offset is corrected) where $\sigma_{x,y}^{(mag)}$ are the horizontal and vertical RMS values of mechanical vibrations of magnets.
Other more realistic models (of type ATL for instance [4]) have been also studied by using directly the numerical coefficients $a^{(1)}_{i_\alpha}$ and $a^{(2)}_{i_\alpha, j_\beta}$ computed by our code FFSER2.

5 CONCLUSION

Compared to conventional tracking methods, this approach presents then the following two advantages: its short running time (FFSER2 takes less then five minutes to scan the whole TESLA FFS containing about 50 magnets) and mainly the possibility to study in a systematic and synthetic way the sensitivity to errors of any beam transfer line by analyzing their impacts on the transfer map and on the beam matrix. It has been successfully applied to specify the tolerances of the final focus systems of TESLA and SBLC [5] as well as to re-compute the ones of SLC [6].

6 REFERENCES

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