STUDY OF THE DYNAMIC APERTURE OF THE 4D QUADRATIC MAP USING INVARIANT MANIFOLDS

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Abstract

A 4D quadratic map can be used to represent the transfer map of a FODO cell with a sextupolar nonlinearity in the kick approximation. This map describes the transverse betatronic motion of a charged particle in a circular accelerator. The dynamic aperture of such a mapping is analysed, i.e. the domain in phase space where stable motion occurs, as a function of the linear tunes. Starting from the study of the stability properties of the fixed points of low period (one or two), it is shown that the dynamic aperture is related to the invariant manifolds emanating from unstable points. This represents a generalisation of a similar result obtained for generic two-dimensional symplectic maps.

1 INTRODUCTION

The evaluation of the dynamic aperture (DA), which is the volume in phase space where stable motion occurs, of four-dimensional dynamical systems, is based on CPU-time intense simulations which do produce numerical information, but cannot provide a deep insight into the phenomena which determine the DA.

In this paper, a new method to estimate the stability domain of a four-dimensional system is presented. It is based on the properties of low-order unstable fixed points and it is a generalisation of a technique successfully applied to two-dimensional Hamiltonian systems [1, 2, 3]. The method is based on the construction of the invariant manifolds emanating from the unstable fixed points (such manifolds are equivalent to the separatrices for continuous-time Hamiltonian systems) of low period. Thanks to the phenomenon of homoclinic/heteroclinic intersections, these manifolds form a dense network in phase space which extends from the outer, unstable part of phase space to the DA. Hence from the knowledge of the unstable fixed points and the invariant manifolds it is possible to deduce the stability domain of the dynamical system.

In the paper a proof-of-principle of this method is presented using a 4D quadratic map, the generalisation of the well-known 2D area-preserving Hénon map. The model is analysed and its DA is evaluated using the standard numerical methods [4]. Using a simplified technique, the invariant manifolds are numerically constructed and it is shown how the DA can be deduced.

2 THE MODEL

The starting point of the analysis is the transfer map $\mathcal{M}$ of a FODO cell with a sextupole magnet represented by a sextupolar kick. In case of absence of linear coupling, the linear part $\mathcal{L}$ of the transfer function can be written as a block diagonal matrix

$$\mathcal{L} = \begin{pmatrix} \mathcal{M}_x & 0 \\ 0 & \mathcal{M}_y \end{pmatrix}, \tag{1}$$

while the nonlinear kick can be expressed as a polynomial function of degree two, namely

$$\begin{aligned} x_1 &= x_0 \\ x_1' &= x_0' + \mathcal{K}_2(x_0^2 - y_0^2) \\ y_1 &= y_0 \\ y_1' &= y_0' - 2\mathcal{K}_2 x_0 y_0, \end{aligned}$$

where $\mathcal{K}_2$ is the integrated sextupolar gradient, given by

$$\mathcal{K}_2 = \int_L \frac{1}{B_0\rho_0} \frac{\partial^2 B_y}{\partial x^2}\bigg|_{(0,0;s)} d\,s. \tag{2}$$

The global transfer map is given by the composition of $\mathcal{L}$ and the kick

$$\begin{pmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} \mathcal{M}_x & 0 \\ 0 & \mathcal{M}_y \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' + \mathcal{K}_2\,(x_0^2 - y_0^2) \\ y_0 \\ y_0' - 2\mathcal{K}_2\, x_0\, y_0 \end{pmatrix}. \tag{3}$$

Eq. (3) can be cast in a different form by means of the standard transformation to normalised coordinates (Courant-Snyder variables), together with a rescaling of the physical variables by the quantity $\beta_x^{3/2}\mathcal{K}_2$

$$\begin{pmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \end{pmatrix} = \mathcal{R}(\omega_1,\omega_2) \begin{pmatrix} x_0 \\ x_0' + (x_0^2 - \beta\, y_0^2) \\ y_0 \\ y_0' + 2\beta\, x_0\, y_0 \end{pmatrix}. \tag{4}$$

In the previous equation, $\mathcal{R}(\omega_1,\omega_2)$ is a 4D block diagonal matrix representing a 2D rotation in each phase space plane by an angle $\omega_i = 2\pi\nu_i$, while the dimensionless parameter β represents the ratio of the vertical to the horizontal β-functions at the centre of the sextupole. The transfer function (4) is a symplectic quadratic polynomial map $\mathcal{H}$ and it represents the natural generalisation of the well-known 2D Hénon map [5]. It has two remarkable properties:

P1: In the limit $\beta \to 0$ the two phase planes (x, x') and (y, y') are decoupled. In fact, under this condition, the map reduces to a 2D Hénon map in the (x, x') plane and a linear rotation in the (y, y') plane;

P2: the plane (x, x') is invariant under the action of the 4D map (4).

3 FIXED POINTS

A fixed point of a 4D map $\mathcal{G}$ is a root of the polynomial equation

$$\mathcal{G}(\mathbf{x}) = \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^4. \tag{5}$$

The stability properties of the solutions of Eq. (5), which is actually a system of four nonlinear coupled equations, derive from the linearisation $\mathcal{G}_{\mathrm{L}}$ of the map $\mathcal{G}$ around the fixed point. More precisely, the eigenvalues of $\mathcal{G}_{\mathrm{L}}$ determine whether the fixed point is unstable (also called hyperbolic) or stable (also called elliptic). The classification of the fixed points is similar to the one used for the 2D maps, although the number of possible cases is bigger.

Furthermore, it is possible to define higher-order fixed points (also called cycles or periodic orbits) by replacing $\mathcal{G}$ in Eq. (5) with the nth power of the polynomial map

$$\mathcal{G}^n(\mathbf{x}) = \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^4. \tag{6}$$

As far as the fixed points of the map $\mathcal{H}$ with $n = 1, 2$ are concerned, it is possible to show the following results [6]
R1: Four fixed points exist. One is the origin $(0, 0, 0, 0)$ and the others are

$$x_{2\mathrm{D}} = 2\,\tan\frac{\omega_1}{2} \qquad x'_{2\mathrm{D}} = -x_{2\mathrm{D}}\,\tan\frac{\omega_1}{2}$$
$$y_{2\mathrm{D}} = 0 \qquad y'_{2\mathrm{D}} = 0$$

and

$$x_{\pm} = -\frac{1}{\beta}\tan\frac{\omega_2}{2} \qquad x'_{\pm} = -x_{\pm}\,\tan\frac{\omega_1}{2}$$
$$y_{\pm} = \pm\frac{1}{\beta}\left[\frac{1}{\beta^2}\tan^2\frac{\omega_2}{2} + \frac{2}{\beta^2}\tan\frac{\omega_1}{2}\tan\frac{\omega_2}{2}\right]^{1/2}$$
$$y'_{\pm} = -y_{\pm}\,\tan\frac{\omega_2}{2}.$$

By definition, the origin is always stable. The second fixed point $\mathbf{x}_{2\mathrm{D}}$ is real for all the values of the linear tunes ω_1, ω_2. It coincides with the unique hyperbolic fixed point of the 2D Hénon map [5]: this is a consequence of the property **P2**. In the (x, x') plane this fixed point is always hyperbolic, while in the (y, y') its stability type changes from elliptic to hyperbolic, according to the value of the linear tunes.

The third and fourth fixed points are real provided the following condition is satisfied

$$\frac{1}{\beta^2}\tan^2\frac{\omega_2}{2} + \frac{2}{\beta^2}\tan\frac{\omega_1}{2}\tan\frac{\omega_2}{2} \geq 0. \tag{7}$$

They have the same stability type which varies as a function of the linear tunes and β;
R2: No real fixed point of period two exists for the 4D Hénon map.

4 INVARIANT MANIFOLDS

For a hyperbolic fixed point $\mathbf{x}_{\mathrm{hyp}}$ of a 4D map, the eigenvectors of the linearisation of the map $\mathcal{G}_{\mathrm{L}}$ around $\mathbf{x}_{\mathrm{hyp}}$ define two 2D planes in the 4D phase space, where the motion induced by the linearised map has an expanding or a contracting behaviour.

We can extend these sets to the original non-linear map $\mathcal{G}$, i.e. it is possible to define two manifolds emanating from the unstable fixed point, called $\mathcal{W}^{\mathrm{u}}(\mathbf{x}_{\mathrm{hyp}})$ and $\mathcal{W}^{\mathrm{s}}(\mathbf{x}_{\mathrm{hyp}})$, having the same expanding (superscript u) or contracting (superscript s) behaviour. The eigenvectors of $\mathcal{G}_{\mathrm{L}}$ are tangential to $\mathcal{W}^{\mathrm{u,s}}(\mathbf{x}_{\mathrm{hyp}})$ at the fixed point.

The invariant manifolds have at least the hyperbolic fixed point as intersection. An additional intersection, $\mathbf{x}_{\mathrm{hom}}$, is either called homoclinic or heteroclinic depending on whether the two intersecting manifolds emanate from the same hyperbolic fixed point. Provided the two manifolds are non-tangential at the point $\mathbf{x}_{\mathrm{hom}}$, they will oscillate around each other generating a 2D object which fills most of the phase space of the system (at least the unstable part of the phase space).

It is not a trivial task to numerically construct such a manifold. The problem has been completely solved for 1D invariant manifolds and efficient algorithms have been developed. However, in this case even the straightforward method, based on the iteration of a set of initial conditions chosen on the eigenvectors of the linearised map, provides excellent results [1, 2, 3].

The straightforward generalisation of the previous method to the 4D case, could lead to some problems as the different expansion/contraction rates could stretch one direction more than the others, thus collapsing the reconstructed invariant manifold to an-almost-1D object. Recently, some new methods have been proposed to provide an accurate reconstruction of the 2D invariant manifolds [7, 8].

For the proof-of-principle of the method presented in this paper, the invariant manifolds have been constructed by simply computing the evolution of a set of initial conditions distributed in a 4D neighbourhood of the unstable fixed point of the Hénon map: those initial conditions which do not belong to the invariant manifolds are lost due to the hyperbolic dynamics, while the others allow the reconstruction the manifold.

5 SIMULATION RESULTS

The starting point of this study is the numerical evaluation of the DA of the 4D Hénon map as a function of the linear tunes ν_1, ν_2 and the parameter β. The stability domain $D(N)$ in the transverse four-dimensional phase space is given by the volume containing all the initial conditions that are stable at least for N turns. It has been shown [4] that $D(N)$ can be computed by iterating a set of initial conditions $(x, 0, y, 0)$ with

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r \in [0, R], \quad \theta \in [0, \pi/4[$$

using the following formula

$$D(N) = \left(\int_0^{\pi/2} [\overline{r}(\theta; N)]^4 \sin 2\theta d\theta\right)^{1/4} \tag{8}$$

where $\overline{r}(\theta; N)$ is the average amplitude along the last stable orbit for a given θ [4]. In Fig. 5 the stability domain is shown for the Hénon map computed using $N = 1000$ and $\beta = 1$ as a function of the linear tunes. One can clearly

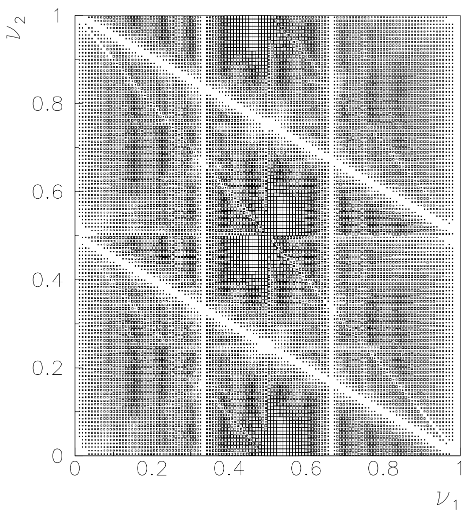

Figure 1: Dynamic aperture $D(N)$ for the 4D Hénon map as a function of the linear tunes ν_1, ν_2 for $N = 1000$ and $\beta = 1$.

see the unstable resonances (third order), where the motion is totally unstable and the DA is zero. Other resonances are also visible, but they have a smaller impact on the DA. In Fig. 5 the DA for $N = 1000$ and $\beta = 0.1$ is shown. The scenario is rather different now: most of the harmful

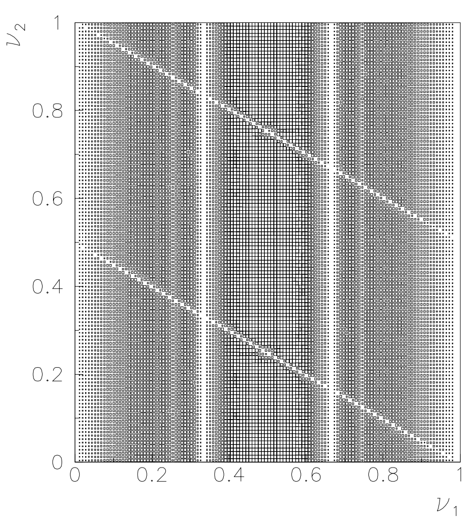

Figure 2: Dynamic aperture $D(N)$ for the 4D Hénon map as a function of the linear tunes ν_1, ν_2 for $N = 1000$ and $\beta = 0.1$.

resonances have disappeared and $D(N)$ shows only two vertical and two diagonal third order resonances. Furthermore, $D(N)$ show almost no sensitivity to ω_2: this is a consequence of the property **P2** as in this case the nonlinear coupling between the two planes is weaker.

Finally Fig. 5 shows the minimum distance of the unstable invariant manifold emanating from the fixed point $\mathbf{x}_{2\mathrm{D}}$ from the origin ($\beta = 0.1$). According to the assumptions made, the invariant manifolds should surround the region where stable motion occurs. Hence, the minimum distance of the manifold from the origin should give an estimate of the DA. In Fig. 5 one clearly detects some features similar

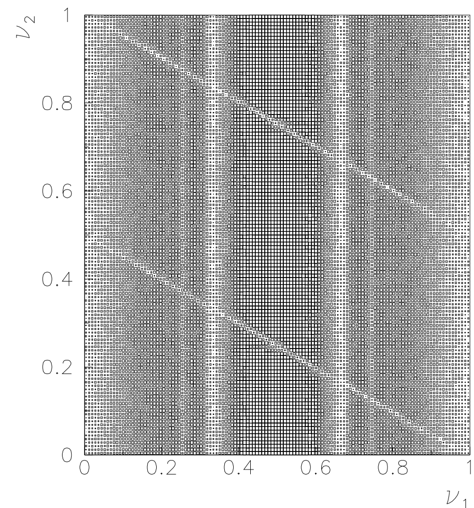

Figure 3: Minimum distance from the origin of $\mathcal{W}^{\mathrm{u}}(\mathbf{x}_{2\mathrm{D}})$ as a function of the linear tunes ν_1, ν_2 for $\beta = 0.1$. The initial conditions are iterated 1000 times.

to those shown by the DA, namely the unstable third order resonance (vertical lines) and also a trace of the coupled third order resonance (diagonal lines).

6 CONCLUSIONS

The numerical simulations showed that the invariant manifolds allow to reproduce the main features of the dynamic aperture. Although the numerical values of the DA and the minimum distance of the invariant manifolds agree within 30-40 %, we are confident that this can be improved by using more sophisticated methods to reconstruct the invariant manifolds.

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