SELF-CONSISTENT BEAM EQUILIBRIUM AND HALO-FREE BEAM TRANSPORT

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Abstract

Intense nonuniform particle beam exhibits strong emittance growth and halo formation in linear focusing channel due to nonlinear space charge forces of the beam (see Fig. 1). This phenomenon limits beam brightness and results in particle losses. The problem is connected with irreversible distortion of phase space volume of the beam in conventional focusing structures due to filamentation in phase space. Emittance growth is accompanied with halo formation in real space which finally results in inevitable particle losses. New approach for solving a self-consistent problem for a matched nonuniform beam in two-dimensional geometry is discussed. Resulting solution is applied to the problem of beam transport, while avoiding emittance growth and halo formation by the use of nonlinear focusing field. Conservation of a beam distribution function is demonstrated via analytical derivations and utilizing particle-in-cell simulation for a beam with a realistic beam distribution.

1 MATCHED BEAM DISTRIBUTION

To prevent emittance growth and halo formation, beam has to be matched with the channel. Beam distribution function, $f\,(x, p_x, y, p_y)$, expressed as a function of Hamiltonian, H, is a constant of motion in a z - uniform, time-independent focusing channel

$$f = f\,(H), \qquad H = \frac{p_x^2 + p_y^2}{2\ m\ \gamma} + q\ U_{ext} + q\ \frac{U_b}{\gamma^{\ 2}}\ , \tag{1}$$

where p_x and p_y are components of transverse particle momentum, q and m are charge and mass of the particles, respectively, γ is a particle energy, U_{ext} is a potential of focusing field, and U_b is a space charge potential of the beam. Matched beam distribution function, Eq. (1), obeys self-consistent set of Vlasov-Poisson equations:

$$\left\{ \begin{array}{l} \dfrac{1}{m\gamma}\left(\dfrac{\partial f}{\partial x}p_x + \dfrac{\partial f}{\partial y}p_y\right) - q\left(\dfrac{\partial f}{\partial p_x}\dfrac{\partial U}{\partial x} + \dfrac{\partial f}{\partial p_y}\dfrac{\partial U}{\partial y}\right) = 0 \\[2ex] \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\,\dfrac{\partial U_b}{\partial r}\right) = -\dfrac{q}{\varepsilon_o}\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,(H)\,dp_x\,dp_y \end{array} \right. , \tag{2}$$

where $U = U_{ext} + U_b\gamma^{-2}$ is a total potential of the structure. In many cases beam distribution function is known, for example, from particle source. The problem is, therefore, to keep given beam distribution function and to prevent undesirable distortion of phase space volume of the beam.

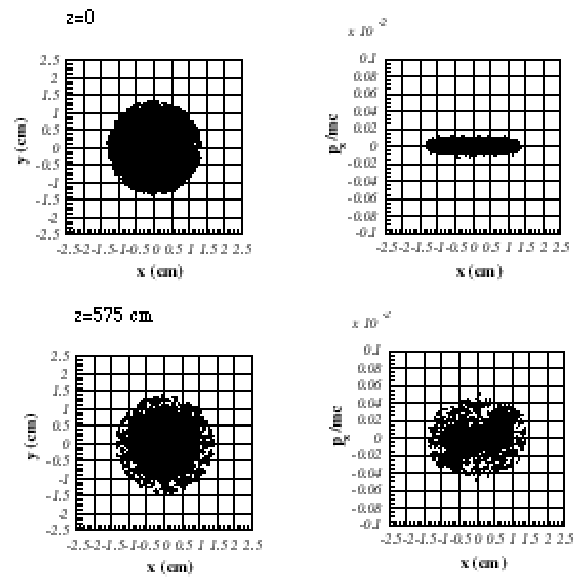

Figure 1. Emittance growth and halo formation of 150 keV, 0.1 A, 0.06 π cm mrad proton beam with truncated at $2.6\sqrt{<x^2>}$ Gaussian distribution in linear focusing channel.

2 BEAM EQUILIBRIUM IN CONTINUOUS FOCUSING CHANNEL

General method to solve the problem, Eq. (2) is to substitute the given beam distribution function into Vlasov's equation and to solve it for unknown total potential of the structure, U [1]. Required focusing field is then found as a difference between the total potential and known space charge potential of the beam $U_{ext} = U - U_b\gamma^{-2}$. The same relationship is valid for electrical field $E_{ext}(r) = E(r) - E_b(r)\,\gamma^{-2}$.

Consider the following class of beam distribution functions

$$f = f\,(T)\ , \qquad T = \frac{p_x^2 + p_y^2}{p_o^2} + G(x, y)\ . \tag{3}$$

Calculation of derivatives of the distribution function give:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial T}\frac{\partial G}{\partial x}\ , \qquad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial T}\frac{\partial G}{\partial y}\ ,$$

$$\frac{\partial f}{\partial p_x} = 2\frac{\partial f}{\partial T}\frac{p_x}{p_o^2} \qquad \frac{\partial f}{\partial p_y} = 2\frac{\partial f}{\partial T}\frac{p_y}{p_o^2} \qquad (4)$$

Substitution of derivatives, Eq. (4), into Vlasov's equation, provides relationship for unknown total potential of the structure:

$$\frac{1}{m\gamma}\left(\frac{\partial G}{\partial x}p_x + \frac{\partial G}{\partial y}p_y\right) - 2q\left(\frac{p_x}{p_o^2}\frac{\partial U}{\partial x} + \frac{p_y}{p_o^2}\frac{\partial U}{\partial y}\right) = 0 . \qquad (5)$$

Particular solution of the equation (5) can be obtained, if components of Eq. (5) are equal termwise:

$$\frac{\partial U}{\partial x} = \frac{p_o^2}{2qm\gamma}\frac{\partial G}{\partial x} , \qquad \frac{\partial U}{\partial y} = \frac{p_o^2}{2qm\gamma}\frac{\partial G}{\partial y} . \qquad (6)$$

In this case the total potential of the structure is proportional to the function G(x,y):

$$U(x,y) = \frac{p_o^2}{2qm\gamma}G(x,y) . \qquad (7)$$

Therefore, given beam distribution function uniquely defines required total potential of the structure.

3 BEAM EQUILIBRIUM WITH ELLIPTICAL SYMMETRY IN PHASE SPACE

Let the function G(x,y) in the beam distribution, Eq. (3), to be a quadratic function of radius, r:

$$G = \frac{x^2 + y^2}{R^2} , \qquad r^2 = x^2 + y^2 . \qquad (8)$$

Equations (3) and (8) define a class of distribution functions with elliptical symmetry in phase space. Particle trajectories of the equilibrium beam are given by family of ellipses:

$$\frac{x^2}{R^2} + \frac{p_x^2}{p_o^2} = \text{const} . \qquad (9)$$

Let us introduce normalized beam emittance $\varepsilon = R\, p_o/(mc)$. Total potential of the structure for this class of distributions from Eq. (7) is also quadratic function of radius

$$U = \frac{mc^2}{q}\frac{1}{\gamma}\left(\frac{\varepsilon}{R}\right)^2\left(\frac{x^2 + y^2}{2R^2}\right) . \qquad (10)$$

Therefore, total field has to be linear function of radius:

$$E = -\frac{mc^2}{qR}\frac{1}{\gamma}\left(\frac{\varepsilon}{R}\right)^2\left(\frac{r}{R}\right) . \qquad (11)$$

This result has a simple explanation. Particles rotate in total linear field along elliptical phase space trajectories, Eq. (9), and beam distribution with elliptical symmetry is, therefore, conserved.

Equilibrium of the KV beam

One of the distributions of this type is the KV distribution [2]

$$f = f_o\,\delta\left(\frac{p_x^2 + p_y^2}{p_o^2} + \frac{x^2 + y^2}{R^2} - T_o\right) . \qquad (12)$$

Distribution, Eq. (12), is projected at configuration space (x,y) as uniformly populated beam of radius R with space charge potential

$$U_b = -\frac{I}{4\pi\varepsilon_o\beta c}\frac{(x^2 + y^2)}{R^2} , \qquad (13)$$

where I is a beam current and β is a longitudinal particle velocity. Required focusing field for the KV beam is a combination of total field, Eq. (10), and space charge field, Eq. (13):

$$U_{ext} = \frac{mc^2}{q}\frac{1}{\gamma}\left(\frac{r}{R}\right)^2\left[\frac{1}{2}\left(\frac{\varepsilon}{R}\right)^2 + \frac{I}{\beta\gamma I_c}\right] , \text{ or} \qquad (14)$$

$$E_{ext} = -\frac{mc^2}{qR}\frac{1}{\gamma}\left(\frac{r}{R}\right)\left[\left(\frac{\varepsilon}{R}\right)^2 + 2\frac{I}{\beta\gamma I_c}\right] , \qquad (15)$$

where $I_c = 4\pi\varepsilon_o mc^3/q = 3.13\cdot10^7(A/Z)$ Amp is characteristic value of beam current.

The same relationships, Eqs. (14) and (15), follow directly from KV envelope equation for the axial - symmetric beam in focusing channel with applied potential $U_{ext} = G_f\,(r^2/2)$

$$\frac{d^2R}{dz^2} + \mu_o^2 R - \frac{\varepsilon^2}{(\beta\gamma)^2 R^3} - \frac{2I}{R\,(\beta\gamma)^3 I_c} = 0 , \qquad (16)$$

where μ_o is a frequency of particle oscillations in a linear focusing channel without space charge forces

$$\mu_o^2 = \frac{q\,G_f}{m\,\gamma\,\beta^2c^2} = \frac{q}{m\,\gamma\,\beta^2c^2}\frac{1}{r}\frac{dU_{ext}}{dr} . \qquad (17)$$

If the KV beam is in equilibrium with external focusing field, beam radius R remains constant and the envelope equation, Eq. (16) gives the same equilibrium condition for KV beam, as Eqs. (14) and (15).

Equilibrium of the Gaussian Beam

Let us check, how equilibrium condition is changed for non-uniform beam distribution with elliptical symmetry [1]. Consider beam with Gaussian distribution function

$$f = f_o \exp\left(-2\frac{p_x^2 + p_y^2}{p_o^2} - 2\frac{x^2 + y^2}{R^2}\right) . \qquad (18)$$

Space charge field of the Gaussian beam is given by

$$E_b = \frac{I}{2\pi\,\varepsilon_o\,\beta\,c}\frac{1}{r}\left[1 - \exp\left(-2\frac{r^2}{R^2}\right)\right] . \qquad (19)$$

Equilibrium condition for Gaussian beam is the combination of the total field, Eq. (11) and space charge field, Eq. (19):

$$E_{ext}= - \frac{mc^2}{q\,R}\frac{1}{\gamma}\frac{r}{R}\left[\frac{\varepsilon^2}{R^2} + 4\frac{I}{\beta\gamma I_c}\frac{(1-\exp(-2r^2/R^2))}{2(r^2/R^2)}\right]. \quad (20)$$

Condition to maintain Gaussian beam, Eq. (20), is different from that of KV beam, Eq. (15). For non-uniform beam external focusing field has to be nonlinear function of radius.

4 BEAM EQUILIBRIUM WITHOUT ELLIPTICAL SYMMETRY IN PHASE SPACE

Consider another class of particle distribution with function G(x, y) in Eq. (3) as

$$G = \frac{(x^2 + y^2)^2}{2R_o^4} . \quad (21)$$

Example of this kind of beam distribution is

$$f = f_o \exp \left(- 2\frac{p_x^2 + p_y^2}{p_o^2} - \frac{(x^2+ y^2)^2}{R_o^4} \right) . \quad (22)$$

Phase space trajectories of the equilibrium beam in this case are not ellipses anymore. According to Eq. (7), total potential of the structure for this class of particle distributions is

$$U(x,y) = \frac{p_o^2}{qm\gamma}\frac{(x^2 + y^2)^2}{4\,R_o^4} . \quad (23)$$

Total field of the structure is

$$E = - \frac{1}{\gamma}\frac{mc^2}{q\,R_o}\left(\frac{\varepsilon^*}{R_o}\right)^2\left(\frac{r^3}{R_o^3}\right) , \quad (24)$$

where $\varepsilon^* = R_o p_o/(mc)$ is an effective normalized beam emittance. In contrast with distribution functions with elliptical symmetry, total field, Eq. (24) is not a linear function of radius, but is an essentially nonlinear function of radius $\sim r^3$.

Space charge density of the beam is attained after integration of beam distribution function over particle momentum:

$$\rho(r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\, dp_x\, dp_y = \rho_o \exp \left(- \frac{r^4}{R_o^4}\right) . \quad (25)$$

Space charge field of the beam is obtained from Poisson's equation

$$E_b = \frac{1}{\varepsilon_o r}\int_o^r \rho(r')\, r'\, dr' = \frac{I}{2\pi\varepsilon_o\beta c\, r}\ \mathrm{erf}\left(\frac{r^2}{R_o^2}\right) . \quad (26)$$

Combination of the total field, Eq. (24), and space charge field, Eq. (26), gives the expression for the required focusing field of the structure to maintain beam distribution:

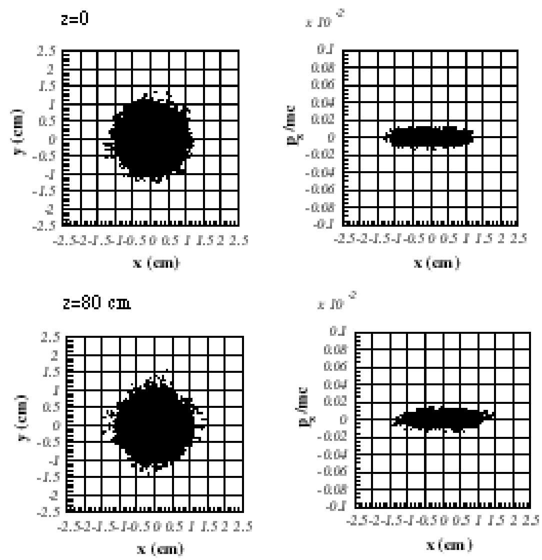

Figure 2. Conservation of 150 keV, 0.5A, 0.07 π cm mrad proton beam with distribution, Eq. (22), in nonlinear focusing field, Eq. (27).

$$E_{ext}= - \frac{1}{\gamma}\frac{mc^2}{q\,R_o}\left[\left(\frac{\varepsilon^*}{R_o}\right)^2\left(\frac{r^3}{R_o^3}\right) + \frac{2I}{I_c\,\beta\gamma}\left(\frac{R_o}{r}\right) \mathrm{erf}\left(\frac{r^2}{R_o^2}\right)\right] . \quad (27)$$

In Fig. 2 results of numerical study of high-brightness beam transport with distribution function, Eq.(22), in focusing field, Eq. (27), are presented. As seen, beam remains in equilibrium in contrast with mismatched beam, presented in Fig. 1.

Important point is stability of beam equilibrium in nonlinear focusing field. Sufficient condition for stability is given by Newcomb-Gardner theorem [3], which states, that monotonically decreasing equilibrium distribution function of Hamiltonian $\partial f/\partial H < 0$ is stable with respect to perturbations. Distributions, Eqs. (18) and (22) as well as most of realistic beam distributions satisfy stability condition.

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