HIGH-ORDER OPTICS WITH SPACE-CHARGE: ANALYTICAL APPROACH

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Abstract

This paper presents an analytical description of the beam dynamics with space charge. The suggested approach is based on the Lie algebraic methods in the matrix formalism. The nonlinear operator differential equation for the corresponding Lie transformation is solved using a convergent recurrent procedure. On each elementary interval the required matrices are calculated in the symbolic form using computer algebra codes. Such calculations are made for some models for beam density distributions. These model distributions can be used for approximation of real beam distributions. The corresponding software is compact and flexible.

1 INTRODUCTION

Many approaches to the space charge modeling problem are known. The basic difficulties which arise from this problem is a nonlinearity of corresponding motion equations even for simplest cases. In this paper some evolution problems of long beams with an elliptical cross--section in transverse phase space are discussed. As examples some special forms of phase space distribution function are described [1]. For these distribution functions the Ferrer's integrals technique is used. It allows to obtain motion equations in symbolic forms as expansions in series. These equations are written in a matrix form. The truncated matrix equation (up to a certain order) are solved with the use of the matrix formalism for Lie algebraic tools [2]. Nonlinear nature of these equations leads to a necessity to use the successive approximations method. Obtained convergence conditions and algorithms give opportunity to estimate a current step value in advance and to create necessary software for modeling. Moreover the suggested approach allows to use object--oriented ideology and to realize the dynamic modeling paradigm [3, 4].

2 A SPACE CHARGE DESCRIPTION

2.1 The Initial Space Charge Distributions

The space charge distribution in the phase space is described with the help of a set of phase-space distribution functions [1]. Note that in general a phase-space distribution function $f_0(X) = f(X, s_0)$ can be written in the form $f_0(X) = \sum_{k=0}^{\infty} f_k^0 X^{[k]}$ or in the case of elliptical symmetry

$$f_0(X) = \sum_{k=0}^{\infty} a_k^0 \kappa^{2k} = \sum_{k=0}^{\infty} a_k^0 (X^{[k]}) * A_0^{\{k\}} X^{[k]}, \quad (1)$$

Where $X^{[k]} = \underbrace{X \otimes \cdot \ldots \cdot \otimes X}_{k-times}$ is the Kronecker power of the phase vector $X = (x, p_x, y, p_y)^*$, $\dim X^{[k]} = \begin{pmatrix} k+3 \\ k \end{pmatrix}$, $A_0^{\{k\}}$ is the symmetrical Kronecker power of the initial form matrix A_0: $\kappa^2 = X^* A_0 X$:

$$(A^{\{k\}})_{il} = b_i(k)(A_0^{[k]})_{il}, \; i,l = \overline{1, d[n,k]},$$

$$d[n,k] = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$$

and $b_i(k)$ are the polynomial coefficients of k-th order, which can be calculated from the known expression $k!/k_1!\ldots k_n!$ using the lexicographic order

$k_1 \succ k_2 \succ \ldots \succ k_n$

2.2 The Self-Field of the Space Charge

Using the Ferrers's integrals technique we can calculate the desired potential for the beam in the form

$$V = -\frac{\pi ab}{\varepsilon_0} \int_0^{\infty} \Phi(\kappa_1^2(u)) \frac{du}{\Delta(u)}$$

when $\Phi(\kappa^2) = \int_0^{\kappa^2} \rho(t)dt$, $\Delta^2(u) = (a^2+u)(b^2+u)$, $\kappa_1^2(u) = \xi^2/(a^2+u) + \eta^2/(b^2+u)$ where ξ, η some local coordinates in which the phase ellipsoid is the canonical one. After some transformations we obtain the components of the vector of self-electrical field

$$E_\xi = \frac{2\pi ab}{\varepsilon_0} \rho_0 \int_0^{\infty} \left\{ \frac{\xi}{a^2+u} \Phi(\kappa_1^2(u)) \right\} \frac{du}{\Delta u},$$

$$E_\eta = \frac{2\pi ab}{\varepsilon_0} \rho \int_0^{\infty} \left\{ \frac{\xi}{b^2+u} \Phi(\kappa_1^2(u)) \right\} \frac{du}{\Delta u}.$$

and for our particle density distributions we can write

$$E_{\xi,\eta} = E^0_{\xi,\eta} + \Delta E_{\xi,\eta}, \qquad E^0_\xi = \frac{4\pi\rho_0}{\varepsilon_0} \frac{ab}{a(a+b)} \xi,$$

$$E_\eta^0 = \frac{4\pi\rho_0}{\varepsilon_0}\frac{ab}{b(a+b)}\eta$$

The values of ΔE_ξ, ΔE_ηcan be easily calculated with the help of REDUCE (or MATHEMATICA) codes. Note that for the K-V distribution we have ΔE_ξ, = ΔE_η =0. Besides, if the arbitrary distribution $\rho(x,y) = \rho_0\Phi(\kappa_1^2)$ is a polynomial arbitrary distribution of *n*-th order with respect to the variables κ_1^2 the functions and ΔE_ξ, and ΔE_η are polynomials of $(2n+1)$-th order with respect to the variables ξ and η. Then we can return to the coordinates *x, y* referenced to a desired system.

3 THE MOTION EQUATION

3.1 The Motion Equation for a Single Particle

As an example we consider the transport system with the rectilinear optical axis (there are no dipole magnets). For a nonbunched beam (the longitudinal self-electric field is missing) the motion equation for single particle can be written in the form

$$x'' = \frac{q}{p}T\left\{y'B_S - (1-x'^2)B_y + x'y'B_x + T\frac{E_x}{\beta c\gamma^3}\right\},$$

$$y'' = -\frac{q}{p}T\left\{x'B_S - (1-y'^2)B_x + x'y'B_y - T\frac{E_y}{\beta c\gamma^3}\right\},$$

where, $T = \sqrt{1+x'^2+y'^2}$, $' = d/ds$. Note that the self-magnetic field is studied with the help of the coefficient $1/\gamma^3$.

These motion equations can be rewritten in the following matrix form

$$\frac{dX}{ds} = \sum_{k=1}^{\infty}\left\{\mathbf{P}_{ext}^{1k}(s) + \mathbf{P}_{self}^{1k}(s)\right\}X^{[k]}$$

The matrices $\mathbf{P}_{ext}^{1k}$ and $\mathbf{P}_{self}^{1k}$ describe the external and self-fields correspondingly. As an example the corresponding motion equations for a quadrupole lens system in the presence of space-charge up to third order on phase space variables are considered in [1, 6].

3.2 Transfer Map in the Presence of the Space-Charge

The transfer map approach has been very useful for studying beam dynamics without space-charge. In this case we can write

$$\mathcal{M}: X_0 \to X = \mathcal{M}(s|s_0)\cdot X_0,$$

where $\mathcal{M}$ is called the transfer map (or Lie map) between moments s_0 and *s*. If the beam is an ensemble of noninteracting particles then the map $\mathcal{M}(s|s_0)$ depend only on control system parameters. In the frame of the matrix formalism [2] we can represent this map in the form

$$X = \mathcal{M}(s|s_0)\cdot X_0 = \sum_{k=1}^{\infty}\mathbf{M}^{1k}(s|s_0)X_0^{[k]}, \qquad (2)$$

where $\mathbf{M}^{1k}$are matrices which can be calculated with the help of the matrix formalism tools

According to the previous sections our beam has the elliptical cross-section. So let $\mathfrak{M}_0$ be an initial phase set occupied by particles:

$$\mathfrak{M}_0 = \left\{X_0 :\ X_0^*\mathbf{A}_0^{11}X_0 \le 1\right\}.$$

Let $\mathfrak{S}_0^{11} = (\mathbf{A}_0^{11})^{-1}$. The other definition of the $\mathfrak{S}$-matrix (so-called rms-envelope matrix) is following

$$\tilde{\mathfrak{S}}_0^{11} = \int_{\mathfrak{M}_0} f_0(X)XX^*dX,$$

where f_0 is an initial phase space distribution. The similar formula can be used for the definition

$$\tilde{\mathfrak{S}}_0^{ik} = \int_{\mathfrak{M}_0} f_0(X)X^{[i]}(X^{[k]})^*dX.$$

These definitions allow the beam dynamics to be investigated in the terms of envelopes. Indeed, according to the matrix formalism for both cases of the definition of the $\mathfrak{S}$-matrix we have

$$\mathfrak{S}^{11}(s) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\mathbf{M}^{1k}\ \mathfrak{S}_0^{kj}(\mathbf{M}^{1j})^*.$$

In the presence of the space-charge the Lie map will depend on the parameters of the beam too. In this case the operator equation for the map $\mathcal{M}$ is a nonlinear equation. The solving this equation requires the another approach. In this paper we propose the method of step-by-step approximations. The basic idea of this method in our case is to calculate the envelope matrix according to the following algorithm (compare with [5]).

Step 0:

1) Choose the initial model distribution function $f_0(x)$ according to the Eq.(1) and the order of our approximation equal to *N*.

2) Define the set of matrices $\mathfrak{S}_0^{ik}$ according to the corresponding equations and then form the initial *nonlinear* envelope matrix $\mathfrak{S}_0^N$:

$$\mathfrak{S}_0^N = \left\{\mathfrak{S}_0^{ik}\right\}_{i,k=\overline{1,N}}.$$

3) Define the sequence of subintervals $I_i = \left[s_i, s_{i+1}\right)$,

$$\bigcup_{i=0}^{m} I_i = I_{tot} = \left[s_0, s_*\right],\ i = \overline{0,m},\ s_m = s_*$$

Step 1:

Calculate the auxiliary envelope matrices by

$$\mathfrak{S}_1^{ik}(s_0) = \alpha\,\mathfrak{S}_0^{ik}(s_0) + (1-\alpha)\mathfrak{S}_0^{ik}(s_1),\ \ 0<\alpha<1,$$

$$\mathfrak{S}_0^{ik}(s_1) = \sum_{j,l=1}^{N}\mathbf{M}_1^{ij}\,\mathfrak{S}_0^{jl}(s_0)(\mathbf{M}_1^{lk})^*,$$

$\mathbf{M}_1^{ij} = \mathbf{M}^{ij}(s \mid s_0; \{\mathfrak{S}_0^{kl}\})$ is the block matrices calculated according to the matrix formalism by constant matrices $\mathfrak{S}_0^{kj}$ (self-consistent motion equations or self-consistent Hamiltonian functions), see the Eq.(2).

Step 2:

If $\|\mathfrak{S}_1^{11}(s_1) - \mathfrak{S}_0^{11}(s_1)\| < \varepsilon$ then go to finish else change s_1 on s_2 and s_0 on s_1 and return to the first step.

Note that in the frameworks of the matrix formalism we can also calculate the particle-density distribution function. According to the Eq.(2) and the properties of the Kronecker product we can write the transformation of $X^{[k]}$:

$$X^{[k]} = \left(\sum_{j=1}^{\infty} M^{1j} X_0^{[j]} \right)^{[k]} = \sum_{j=1}^{\infty} M^{kj} X_0^{[j]}$$

$$M^{ml} = \sum_{\substack{k_1 + \ldots + k_n = 1 \\ k_i \geq 1}} \bigotimes_{i=1}^{m} M^{1k_i}, l \geq m$$

From the known properties of Lie maps we can write for an arbitrary function of an initial distribution $f_0(X) = f(X, s_0) : f(X, s) = f_0(\mathcal{M}^{-1}(s|s_0) \cdot X)$. In our case we have $\mathcal{M}^{-1}(s|s_0) \cdot X = \sum_{k=0}^{\infty} T^{1k}(s|s_0) X^{[k]}$

where $\mathbf{T}^{10} = -\mathbf{M}^{10}$, $\mathbf{T}^{11} = (\mathbf{M}^{11})^{-1}$ and other matrices $\mathbf{T}^{1k}$ for $k > 1$ can be calculated with the help of the recurrent generalized Gauss's algorithm using the matrices $\mathbf{M}^{1k}$. It is worthy to note that according to this algorithm one should inverse only the matrix $\mathbf{M}^{11}$ and then use only matrix operations (with the Kronecker operations extension) for calculation the necessary matrices $\mathbf{T}^{1k}$ up to the desired order. So after some calculations we can obtain the following equation (see the Eq.(1))

$$f(X, s) = f_0(\mathcal{M}^{-1} \cdot X) =$$
$$a_0^0 \sum_{k=1}^{\infty} a_k^0 (\mathcal{M}^{-1} \cdot X^{[k]})^* \mathbf{A}_0^{\{k\}} (\mathcal{M}^{-1} \cdot X^{[k]}) =$$
$$a_0^0 + \sum_{k=1}^{\infty} a_k^0 \sum_{l=k}^{\infty} \sum_{j=k}^{\infty} (X^{[l]})^* \mathbf{B}_k^{lj} X^{[j]},$$

$$\mathbf{B}_k^{lj} = (\mathbf{T}^{kl})^* \mathbf{A}_0^{\{k\}} \mathbf{T}^{kj}.$$

For the convergence of our approximations methods for envelope matrices the following condition has to be satisfied

$$\|\mathfrak{S}_k^{11} - \mathfrak{S}_{k-1}^{11}\| \leq \beta \|\mathfrak{S}_{k-1}^{11} - \mathfrak{S}_{k-2}^{11}\|$$

for $\beta < 1$. The constant of this method β can be calculated as a function of the initial beam characteristics and the transport system parameters. The condition $\beta < 1$ allows the limitations on the step values $|s_k - s_{k-1}| = \Delta s_k$ to be calculated which guarantee the fulfilment of the inequality $\beta < 1$. The preliminary calculations showed that it is sufficient to use two - four steps of this iteration process for some value of Δs_k. Note that the convergence condition does not depend on the parameter α.

4 SOME NUMERICAL RESULTS

The above discussed approach was used for some practical problems: the influence of the space--charge forces on the optimal characteristics of the microprobe system [6] and the halo formation problem [7]. The corresponding computer experiments show the advantages and flexible properties of the suggested approach. The symbolic formulae for some models of the space--charge distribution allow to increase effectiveness of the calculations. Moreover this approach gives the powerful tools for the deep investigations of the space--charge problem both in short focusing systems and in circular accelerators and storage rings.

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