THEORY OF A TRANSVERSE FEEDBACK SYSTEM WITH A NONLINEAR TRANSFERFUNCTION

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Abstract

The theory of a transverse feedback system with a nonlinear transfer function is described. For this feedback loop the dependence between the kick and the beam deviation at the pick-up location is a nonlinear function. The beam dynamic equation of the transverse coherent motion for deviation from the closed orbit has been obtained. This nonlinear equation has been solved using approximation procedure. Damping time and asymptotic behaviour of the beam oscillation amplitude are analysed for different parameters of the nonlinear transfer function.

1 INTRODUCTION

Transverse feedback systems (TFS) are used in synchrotrons to damp the coherent transverse beam oscillations. In these systems the kicker (DK) corrects the beam angle according to the beam deviation from the closed orbit in the pick-up (PU) location at every turn. TFS have been widely used to suppress resistive wall instability and to provide a beam oscillation amplitude decrease after injection. Most TFS use digital electronics for signal processing in the feedback loop [1]. Hence, its transfer function has a quasi-linear character. On the SPS, the "bang-bang" regime was realized to increase the efficiency of TFS [2]. Its transfer function consists of a linear part for small amplitude oscillations and a high fixed level gain for large oscillations. For modern hadron accelerators, there is a proposal of TFS with the so-called "logical mode" of damping [3]. Its transfer function is a step function with two jumps. Thus, the transfer characteristics of these TFS are nonlinear. There are only estimations for its damping parameters. For example, in [3] a numerical simulation was used for estimation of damping time.

This paper is based on analytical description for beam dynamics. The nonlinear equation of the transverse coherent motion for deviation from the closed orbit is obtained. Approximate solution of this equation is found using the Krylov–Bogoliubov method [4]. Damping time and asymptotic behaviour of the beam oscillation amplitude are analysed for different parameters of the nonlinear transfer function. All results are obtained for feedback description when instability is neglected.

2 THEORY

2.1 Basic Equation

The equation of the transverse coherent motion for the particle deviation from the closed orbit $x[n,s]$ at the n-th turn can be written as

$$\left[\frac{d^2}{ds^2} + K(s)\right] x[n,s] = \Delta x'[n]\; \delta(s - s_K), \, \psi \qquad (1)$$

where $K(s)$ is a focusing strength and δ is Dirac's delta function. A localized kick is determined by $\Delta x'$.

In (1) all effects, which occur due to resistive wall instability, are neglected and the matrix method becomes suitable for the beam motion description. Let the column matrix $\widehat{X}[n,s]$ determine the bunch state at the n-th turn at point s of the circumference C_0. The first element of this matrix equals the beam deviation $x[n,s]$ from the closed orbit and the second one is the beam angle $x'[n,s]$. After a short DK the x' value of the bunch is changed by $\Delta x'[n,s_K]$, while deviation remains the same as before the DK at point s_K^-. Hence, after DK at point s_K^+, the beam state is

$$\widehat{X}[n,s_K^+] = \widehat{X}[n,s_K^-] + \Delta\widehat{X}[n,s_K],$$

where the first element of column matrix $\Delta\widehat{X}[n,s_K]$ is zero and the second one equals $\Delta x'[n,s_K]$.

Let us introduce the unperturbed revolution matrix $\widehat{M_0}$ from point s_P of the PU location to point $s_P + C_0$ and the transfer matrix $\widehat{M_1}$ from point s_K of the DK location to point $s_P + C_0$. Then at the PU location at the $(n+1)$-th turn the beam state is

$$\widehat{X}[n+1,s_P] = \widehat{M_0}\widehat{X}[n,s_P] + \widehat{M_1}\Delta\widehat{X}[n,s_K].\psi \qquad (2)$$

After not too difficult transformations, a difference equation of the second order can be obtained for $x[n,s_p]$ from the matrix equation (2):

$$\begin{aligned} x[n+2,s_P] - 2x[n+1,s_P]\cos\mu + x[n,s_P] \\ = \sqrt{\beta_P\beta_K}\;\Delta x'[n+1,s_K]\sin(\mu-\eta) \\ + \sqrt{\beta_P\beta_K}\;\Delta x'[n,s_K]\sin\eta\;,\;\psi \end{aligned} \qquad (3)$$

where β_P and β_K are the transverse betatron amplitude functions in the PU and DK locations, $\mu = 2\pi Q$ is a betatron phase advance per revolution in transverse plane, Q is the number of unperturbed betatron oscillations per revolution, and η is the betatron phase advance from PU to DK.

Equation (3) fully describes the particle transverse dynamics in an accelerator with the feedback system considered.

2.2 Transfer Function

As rule, a transfer function for a feedback loop is a linear one. In this case the $\Delta x'[n, s_K]$ value is assumed to be proportional to the beam deviation $x[n, s_P]$ in the pick-up location:

$$\Delta x'[n, s_K] = \frac{g}{\sqrt{\beta_P \beta_K}}\, x[n, s_P], \tag{4}$$

where g is the gain of feedback loop. Substituting for $\Delta x'$ from (4) into (3), we obtain

$$\begin{aligned} &x[n+2, s_P] - 2x[n+1, s_P]\cos\mu + x[n, s_P] = \\ &= g\, x[n+1, s_P]\sin(\mu - \eta) + g\, x[n, s_P]\sin\eta\,. \end{aligned} \tag{5}$$

Equation (5) is a linear equation. It can be solved using standard methods, for example, $Z-$transform [5]:

$$\begin{aligned} \widetilde{\mathbf{x}}(z) &= \sum_{n=0}^{\infty} x[n, s] z^{-n}\,; \\ x[n, s] &= \sum_k \mathrm{Res}\left[\widetilde{\mathbf{x}}(z_k) z_k^{n-1}\right]. \end{aligned} \tag{6}$$

The approximate solution of (5) for small g is

$$x[n, s_P] \simeq a_0 \exp\left(\frac{g}{2}\, n \sin\eta\right)\cos(\mu n + \phi_0), \tag{7}$$

where a_0 and ϕ_0 are constants depending on initial conditions.

If transfer function is nonlinear, then the kick value depends on odd powers of x only. Therefore, for $\Delta x'$ we can write:

$$\sqrt{\beta_P \beta_K}\, \Delta x'[n, s_K] = g\, f(x[n, s_P])\,; \tag{8}$$

$$f(x) = x - g_3 x^3 - g_5 x^5 - \ldots\,. \tag{9}$$

Substituting for $\Delta x'$ from (8) into (3) yields

$$\begin{aligned} &x[n+2, s_P] - 2x[n+1, s_P]\cos\mu + x[n, s_P] \\ &\quad = g\, f(x[n+1, s_P])\sin(\mu - \eta) \\ &\qquad + g\, f(x[n, s_P])\sin\eta\,, \end{aligned} \tag{10}$$

Equation (10) is a basic equation for studying beam dynamics with nonlinear transfer function for a feedback loop. This equation is good for numerical calculations and is convenient for analytical work.

Nonlinear transfer functions mentioned in the introduction can be provided with combination of g_i-values in (9). The analytical results are further shown only for a cubic term in order to simplify final expressions. It will be also supposed that the phase advance η from the PU to the DK is equal to an odd number of $\pi/2$ radians. Hence $\sin\eta = 1$ and the best damping is realised.

2.3 Solution (First Approximation)

The gain g in (10) for feedback realized is a small value. Normally, $g \approx 0.01$ for instability damper systems and $g \approx 0.1$ for damping of injection errors. Since g is small, equation (10) is weakly nonlinear, and a number of perturbation methods are available for the determination of approximate solutions of this equation. The Krylov–Bogoliubov method [4] is used bellow.

When $g = 0$, the solution of (10) can be written as

$$\begin{aligned} x[n, s_P] &= a\cos(\mu n + \phi) = a\cos\psi_n\,, \\ \psi_n &= \mu n + \phi\,, \end{aligned} \tag{11}$$

where a and ϕ are constants. When $g \neq 0$, the solution of (10) can still be expressed in the form (11), provided that a and ϕ are considered to be functions of n rather than constants.

In accordance with the Krylov–Bogoliubov method, the solution of (10) can be written as a series of the form

$$x[n, s_P] = a_n \cos\psi_n + \sum_{m=1}^{\infty} g^m \xi_m(a_n, \psi_n)\,, \tag{12}$$

where ξ_i is unknown functions of full amplitude a_n and periodical functions of ψ_n. Functions ξ_i are small corrections of the main harmonic $a_n \cos\psi_n$. The order of these corrections is given by small parameter g. The amplitude and phase are the functions of a_n. Hence, for their derivations we can write:

$$\frac{da_n}{dn} = g f_1(a_n) + g^2 f_2(a_n) + \ldots\,, \tag{13}$$

$$\frac{d\psi_n}{dn} = \mu + g\chi_1(a_n) + g^2\chi_2(a_n) + \ldots\,. \tag{14}$$

Functions ξ_m as the periodical functions of ψ_n can be expanded into the Fourier series:

$$\begin{aligned} \xi_m(a_n, \psi_n) &= \nu_{m0}(a_n) \\ &+ \sum_{k=2}^{\infty}[\nu_{mk}(a_n)\cos k\psi_n + \gamma_{mk}(a_n)\sin k\psi_n]\,, \end{aligned}$$

where $\nu_{m1} = \gamma_{m1} = 0$, because amplitude a_n is the full amplitude of the main (first) harmonic of oscillations. For the left-hand side of (10) we expand all values into a Taylor series taking into account (13) and (14). The first approximation of these expansions is:

$$\begin{aligned} l.h.s. \simeq\ & g f_1(a_n)[\cos(\psi_n + 2\mu) - \cos\psi_n] \\ & + g\chi_1(a_n) a_n[\sin\psi_n - \sin(\psi_n + 2\mu)] \\ & + g\xi_1(a_n, \psi_{n+2}) - 2g\cos\mu\, \xi_1(a_n, \psi_{n+1}) \\ & + g\xi_1(a_n, \psi_n)\,. \end{aligned} \tag{15}$$

For a first level of approximation, the right-hand side of (10) is determined by zero level of approximation. Substituting for x from (12) into (10) yields

$$\begin{aligned} r.h.s. = &\ g\, f(a_n\cos\psi_n) \\ & - g\, f(a_n\cos(\psi_n + \mu))\cos\mu\,. \end{aligned} \tag{16}$$

Equating coefficients of Fourier series in (15) and (16) yields for main harmonic and for cubic nonlinearity:

$$-f_1(a_n) = \frac{1}{2}\,a_n - \frac{3g_3}{8}\,a_n^3\,; \quad (17)$$

$$\chi_1(a_n) = 0\,. \quad (18)$$

For the high order harmonics, a linear equation is obtained

$$\xi_1(a_n,\psi_{n+2}) - 2\cos\mu\,\xi_1(a_n,\psi_{n+1}) + \xi_1(a_n,\psi_n) = -\frac{g_3}{8}\,a_n^3\left(\cos 3\psi_n - \cos(3\psi_n + 4\mu)\right)\,. \quad (19)$$

This linear equation can be solved using $Z-$transform.

3 RESULTS

Equation (17) yields the amplitude damping rate per turn. In accordance with (13) we can write:

$$da_n = -\frac{g}{2}\left(a_n - \frac{3g_3}{4}\,a_n^3\right)dn\,. \quad (20)$$

Performing the integration in (20) with $g_3 = 0$ we obtain

$$a_n = a_0\exp\left(-\frac{g}{2}\,n\right)\,. \quad (21)$$

This expression for a_n coincides with the well known solution of the linear equation (see formula (7) for $\sin\eta = 1$).

Performing the integration in (20) with $g_3 \neq 0$, we obtain the amplitude dependence on turns:

$$a_n = \frac{a_0\exp(-gn/2)}{\sqrt{1 + (3g_3a_0^2/4)(\exp(-gn) - 1)}}\,. \quad (22)$$

The formula (22) for amplitude dependence coincides with the well known result for amplitude solution of Rayleigh's equation

$$\ddot{x} + \omega_0^2 x = \epsilon(\dot{x} - \lambda\dot{x}^3)\,.$$

Taking into account (18), we get from (14) for the phase of oscillations:

$$\psi_n = \mu n + \phi_0\,. \quad (23)$$

Hence to the first level of approximation, the frequency is not affected by the damping, while the amplitude decreases in accordance with dependence (22).

It is clear from (19), that the cubic kick excites the third harmonic of oscillations:

$$\xi_1 = -\frac{g_3 b}{8}\,a_n^3\sin 3\mu n\,, \quad (24)$$

where constant $b < 1$.

Thus, the first approximation to the solution of (10) is

$$x[n,s_P] = a_n\cos(\mu n + \phi_0) - \frac{gg_3b}{8}\,a_n^3\sin 3\mu n\,, \quad (25)$$

where the amplitude a_n of oscillations is given by (22) and $\mu = 2\pi Q$.

The asymptotic result for linear-to-nonlinear first harmonic amplitude ratio in accordance with (21) and (22) is

$$\lim_{n\to\infty}\frac{a_n(linear)}{a_n(nonlinear)} = \sqrt{1 - \frac{3g_3}{4}\,a_0^2}\,. \quad (26)$$

Therefore, when $g_3 > 0$, the kick value of feedback with a cubic nonlinearity for transfer function is lower than for a linear gain ($g_3 = 0$), and the amplitude decreases faster with time for a linear system. When $g_3 < 0$, the amplitude decreases faster for a nonlinear system. This effect can be used for a faster damping of injection errors.

The same conclusion for instability damping can hardly be true. Indeed, an increment of instability and a decrement of feedback are the effects of linear term forces, and these values can appear as arguments of exponential expressions. But it is clear from (25) and (22) that nonlinear term g_3 is not affected by the exponent terms for a_n at the first level of approximation. Hence, the instability damping by feedback with a nonlinear transfer function should be further studied.

4 CONCLUSION

The general approach demonstrated above can be effectively used for studying TFS with a nonlinear transfer function. It gives analytical approximate solutions to calculate the damping time and other parameters of the particle motion.

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