WAVELET APPROACH TO HAMILTONIAN, CHAOTIC AND QUANTUM CALCULATIONS IN ACCELERATOR PHYSICS

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Abstract

This is the second part of two our papers in which we present applications of wavelet analysis to polynomial approximations for a number of accelerator physics problems. We consider applications of very useful fast wavelet transform technique to calculations in symplectic scale of spaces and to quasiclassical evolution dynamics.

1 INTRODUCTION

This is the second part of two our presentations in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of our results from [1], [2], which is based on approach of two of us from [3], [4] to investigation of nonlinear problems – general, with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum, which are considered in the framework of local(nonlinear) Fourier analysis, or wavelet analysis. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases. Now we consider applications of very useful and powerful method of fast wavelet transform to the problems which appear in nonlinear orbital dynamics in storage rings [5]. The first problem is the explicit calculation of quasiclassical evolution which we consider in section 2 and the second problem, which we consider in section 3, is calculations in (perturbed) Hamiltonian systems in cases when we need to consider multiresolution expansion not in one functional space but in infinite scale of spaces with underlying symplectic structure. In section 4 we consider the key point of this approach which gives useful maximally sparse representation of differential operator that allows us to take into account contribution from each level of resolution.

2 QUASICLASSICAL EVOLUTION

Let we consider classical and quantum dynamics in phase space $\Omega = R^{2m}$ with coordinates (x,ξ) and generated by Hamiltonian $\mathcal{H}(x,\xi) \in C^\infty(\Omega;R)$. If $\Phi_t^{\mathcal{H}}: \Omega \longrightarrow \Omega$ is (classical) flow then time evolution of any bounded classical observable or symbol $b(x,\xi) \in C^\infty(\Omega, R)$ is given by $b_t(x,\xi) = b(\Phi_t^{\mathcal{H}}(x,\xi))$. Let $H = Op^W(\mathcal{H})$ and $B = Op^W(b)$ are the self-adjoint operators or quantum observables in $L^2(R^n)$, representing the Weyl quantization of the symbols $\mathcal{H}, b$ [5]

$$(Bu)(x) = \frac{1}{(2\pi\hbar)^n}\int_{R^{2n}} b\left(\frac{x+y}{2},\xi\right)\cdot e^{i<(x-y),\xi>/\hbar}u(y)dyd\xi,$$

where $u \in S(R^n)$ and $B_t = e^{iHt/\hbar}Be^{-iHt/\hbar}$ be the Heisenberg observable or quantum evolution of the observable B under unitary group generated by H. B_t solves the Heisenberg equation of motion

$$\dot{B}_t = \frac{i}{\hbar}[H, B_t].$$

Let $b_t(x,\xi;\hbar)$ is a symbol of B_t then we have the following equation for it

$$\dot{b}_t = \{\mathcal{H}, b_t\}_M, \qquad (1)$$

with initial condition $b_0(x,\xi,\hbar) = b(x,\xi)$. Here $\{f,g\}_M(x,\xi)$ is the Moyal brackets of the observables $f,g \in C^\infty(R^{2n})$, $\{f,g\}_M(x,\xi) = f\sharp g - g\sharp f$, where $f\sharp g$ is the symbol of the operator product and is presented by the composition of the symbols f,g

$$(f\sharp g)(x,\xi) = \frac{1}{(2\pi\hbar)^{n/2}}\int_{R^{4n}} e^{-i<r,\rho>/\hbar + i<\omega,\tau>/\hbar}\cdot f(x+\omega,\rho+\xi)g(x+r,\tau+\xi)d\rho d\tau dr d\omega.$$

For our problems it is useful that $\{f,g\}_M$ admits the formal expansion in powers of $\hbar$: $\{f,g\}_M(x,\xi) \sim \{f,g\} + 2^{-j}\sum_{|\alpha+\beta|=j\geq 1}(-1)^{|\beta|}\cdot(\partial_\xi^\alpha f D_x^\beta g)\cdot(\partial_\xi^\beta g D_x^\alpha f)$, where $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multi-index, $|\alpha| = \alpha_1+\ldots+\alpha_n$, $D_x = -i\hbar\partial_x$. So, evolution (1) for symbol $b_t(x,\xi;\hbar)$ is

$$\dot{b}_t = \{\mathcal{H}, b_t\} + \frac{1}{2^j}\sum_{|\alpha|+\beta|=j\geq 1}(-1)^{|\beta|}\cdot\hbar^j(\partial_\xi^\alpha\mathcal{H}D_x^\beta b_t)\cdot(\partial_\xi^\beta b_t D_x^\alpha\mathcal{H}). \qquad (2)$$

At $\hbar = 0$ this equation transforms to classical Liouville equation

$$\dot{b}_t = \{\mathcal{H}, b_t\}. \qquad (3)$$

Equation (2) plays a key role in many quantum (semiclassical) problem. We note only the problem of relation between quantum and classical evolutions or how long the evolution of the quantum observables is determined by the corresponding classical one. Our approach to solution of systems (2), (3) is based on our technique from [1]-[4] and very useful linear parametrization for differential operators which we present in section 4.

3 SYMPLECTIC HILBERT SCALES VIA WAVELETS

We can solve many important dynamical problems such that KAM perturbations, spread of energy to higher modes, week turbulence, growths of solutions of Hamiltonian equations only if we consider scales of spaces instead of one functional space. For Hamiltonian system and their perturbations for which we need take into account underlying symplectic structure we need to consider symplectic scales of spaces. So, if $\dot{u}(t) = J\nabla K(u(t))$ is Hamiltonian equation we need wavelet description of symplectic or quasicomplex structure on the level of functional spaces. It is very important that according to [8] Hilbert basis is in the same time a Darboux basis to corresponding symplectic structure. We need to provide Hilbert scale $\{Z_s\}$ with symplectic structure [7], [9]. All what we need is the following. J is a linear operator, $J : Z_\infty \to Z_\infty$, $J(Z_\infty) = Z_\infty$, where $Z_\infty = \cap Z_s$. J determines an isomorphism of scale $\{Z_s\}$ of order $d_J \geq 0$. The operator J with domain of definition Z_∞ is antisymmetric in Z: $< Jz_1, z_2 >_Z = - < z_1, Jz_2 >_Z, z_1, z_2 \in Z_\infty$. Then the triple

$$\{Z, \{Z_s | s \in R\}, \quad \alpha = < \bar{J}dz, dz >\}$$

is symplectic Hilbert scale. So, we may consider any dynamical Hamiltonian problem on functional level. As an example, for KdV equation we have $Z_s = \{u(x) \in H^s(T^1) | \int_0^{2\pi} u(x)dx = 0\}$, $s \in R$, $J = \partial/\partial x$, J is isomorphism of the scale of order one, $\bar{J} = -(J)^{-1}$ is isomorphism of order -1. According to [10] general functional spaces and scales of spaces such as Holder–Zygmund, Triebel–Lizorkin and Sobolev can be characterized through wavelet coefficients or wavelet transforms. As a rule, the faster the wavelet coefficients decay, the more the analyzed function is regular [10]. Most important for us example is the scale of Sobolev spaces. Let $H_k(R^n)$ is the Hilbert space of all distributions with finite norm $||s||^2_{H_k(R^n)} = \int d\xi (1 + |\xi|^2)^{k/2} |\hat{s}(\xi)|^2$. Let us consider wavelet transform

$$W_g f(b, a) = \int_{R^n} dx \frac{1}{a^n} \bar{g}\left(\frac{x-b}{a}\right) f(x),$$

$b \in R^n$, $a > 0$, w.r.t. analyzing wavelet g, which is strictly admissible, i.e.

$$C_{g,g} = \int_0^\infty \frac{da}{a} |\hat{g}(\bar{a}k)|^2 < \infty.$$

Then there is a $c \geq 1$ such that

$$\begin{aligned} c^{-1} ||s||^2_{H_k(R^n)} &\leq \int_{H^n} \frac{dbda}{a} (1 + a^{-2\gamma}) |W_g s(b, a)|^2 \\ &\leq c ||s||^2_{H_k(R^n)} \end{aligned}$$

This shows that localization of the wavelet coefficients at small scale is linked to local regularity.

4 FAST WAVELET TRANSFORM FOR DIFFERENTIAL OPERATORS

Let us consider multiresolution representation $\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots$ (see our other paper from this proceedings for details of wavelet machinery). Let T be an operator $T : L^2(R) \to L^2(R)$, with the kernel $K(x, y)$ and $P_j : L^2(R) \to V_j$ $(j \in Z)$ is projection operators on the subspace V_j corresponding to j level of resolution: $(P_j f)(x) = \sum_k < f, \varphi_{j,k} > \varphi_{j,k}(x)$. Let $Q_j = P_{j-1} - P_j$ is the projection operator on the subspace W_j then we have the following "microscopic or telescopic" representation of operator T which takes into account contributions from each level of resolution from different scales starting with coarsest and ending to finest scales:

$$T = \sum_{j \in Z} (Q_j T Q_j + Q_j T P_j + P_j T Q_j)$$

We remember that this is a result of presence of affine group inside this construction. The non-standard form of operator representation [11] is a representation of an operator T as a chain of triples $T = \{A_j, B_j, \Gamma_j\}_{j \in Z}$, acting on the subspaces V_j and W_j:

$$A_j : W_j \to W_j, B_j : V_j \to W_j, \Gamma_j : W_j \to V_j,$$

where operators $\{A_j, B_j, \Gamma_j\}_{j \in Z}$ are defined as $A_j = Q_j T Q_j, B_j = Q_j T P_j, \Gamma_j = P_j T Q_j$. The operator T admits a recursive definition via $T_j = \begin{pmatrix} A_{j+1} & B_{j+1} \\ \Gamma_{j+1} & T_{j+1} \end{pmatrix}$, where $T_j = P_j T P_j$ and T_j works on $V_j : V_j \to V_j$. It should be noted that operator A_j describes interaction on the scale j independently from other scales, operators B_j, Γ_j describe interaction between the scale j and all coarser scales, the operator T_j is an "averaged" version of T_{j-1}.

The operators A_j, B_j, Γ_j, T_j are represented by matrices $\alpha^j, \beta^j, \gamma^j, s^j$

$$\begin{aligned} \alpha^j_{k,k'} &= \int\int K(x, y) \psi_{j,k}(x) \psi_{j,k'}(y) dx dy \\ \beta^j_{k,k'} &= \int\int K(x, y) \psi_{j,k}(x) \varphi_{j,k'}(y) dx dy \\ \gamma^j_{k,k'} &= \int\int K(x, y) \varphi_{j,k}(x) \psi_{j,k'}(y) dx dy \\ s^j_{k,k'} &= \int\int K(x, y) \varphi_{j,k}(x) \varphi_{j,k'}(y) dx dy \end{aligned} \quad (4)$$

We may compute the non-standard representations of operator d/dx in the wavelet bases by solving a small system of linear algebraical equations. So, we have for objects (4)

$$\alpha^j_{i,\ell} = 2^{-j} \int \psi(2^{-j}x - i) \psi'(2^{-j} - \ell) 2^{-j} dx$$

$$
\begin{aligned}
&= 2^{-j}\alpha_{i-\ell}\\
\beta^j_{i,\ell} &= 2^{-j}\int\psi(2^{-j}x-i)\varphi'(2^{-j}x-\ell)2^{-j}dx\\
&= 2^{-j}\beta_{i-\ell}\\
\gamma^j_{i,\ell} &= 2^{-j}\int\varphi(2^{-j}x-i)\psi'(2^{-j}x-\ell)2^{-j}dx\\
&= 2^{-j}\gamma_{i-\ell},
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_\ell &= \int\psi(x-\ell)\frac{d}{dx}\psi(x)dx\\
\beta_\ell &= \int\psi(x-\ell)\frac{d}{dx}\varphi(x)dx\\
\gamma_\ell &= \int\varphi(x-\ell)\frac{d}{dx}\psi(x)dx
\end{aligned}
$$

then by using refinement equations

$$
\begin{aligned}
\varphi(x) &= \sqrt{2}\sum_{k=0}^{L-1}h_k\varphi(2x-k),\\
\psi(x) &= \sqrt{2}\sum_{k=0}^{L-1}g_k\varphi(2x-k),
\end{aligned}
$$

$g_k=(-1)^k h_{L-k-1}, k=0,\dots,L-1$ we have in terms of filters (h_k,g_k):

$$
\begin{aligned}
\alpha_j &= 2\sum_{k=0}^{L-1}\sum_{k'=0}^{L-1}g_kg_{k'}r_{2i+k-k'},\\
\beta_j &= 2\sum_{k=0}^{L-1}\sum_{k'=0}^{L-1}g_kh_{k'}r_{2i+k-k'},\\
\gamma_i &= 2\sum_{k=0}^{L-1}\sum_{k'=0}^{L-1}h_kg_{k'}r_{2i+k-k'},
\end{aligned}
$$

where $r_\ell=\int\varphi(x-\ell)\frac{d}{dx}\varphi(x)dx, \ell\in Z$. Therefore, the representation of d/dx is completely determined by the coefficients r_ℓ or by representation of d/dx only on the subspace V_0. The coefficients $r_\ell,\ell\in Z$ satisfy the following system of linear algebraical equations

$$
r_\ell=2\left[r_{2l}+\frac{1}{2}\sum_{k=1}^{L/2}a_{2k-1}(r_{2\ell-2k+1}+r_{2\ell+2k-1})\right]
$$

and $\sum_\ell \ell r_\ell=-1$, where $a_{2k-1}=2\sum_{i=0}^{L-2k}h_ih_{i+2k-1}$, $k=1,\dots,L/2$ are the autocorrelation coefficients of the filter H. If a number of vanishing moments $M\geq 2$ then this linear system of equations has a unique solution with finite number of non-zero r_ℓ, $r_\ell\neq 0$ for $-L+2\leq \ell\leq L-2, r_\ell=-r_{-\ell}$. For the representation of operator d^n/dx^n we have the similar reduced linear system of equations. Then finally we have for action of operator $T_j(T_j:V_j\to V_j)$ on sufficiently smooth function f:

$$
(T_jf)(x)=\sum_{k\in Z}(2^{-j}\sum_\ell r_\ell f_{j,k-\ell})\varphi_{j,k}(x),
$$

where $\varphi_{j,k}(x)=2^{-j/2}\varphi(2^{-j}x-k)$ is wavelet basis and

$$
f_{j,k-1}=2^{-j/2}\int f(x)\varphi(2^{-j}x-k+\ell)dx
$$

are wavelet coefficients. So, we have simple linear parametrization of matrix representation of our differential operator in wavelet basis and of the action of this operator on arbitrary vector in our functional space.

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