LAMINAR FLOW IN NON-RELATIVISTIC INTENSE PROTON BEAMS

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Abstract

An approach to the envelope description of laminar non-relativistic particle beams is presented, which describes a new kind of equilibrium flow for strongly space charge dominated proton beams subject to acceleration in RF Linacs. The analysis is based on the extension of the invariant envelope concept, recently introduced in the field of RF photo-injectors[1], to non-relativistic particle beams whose envelope is dominated by coherent plasma oscillations instead of incoherent betatron motion associated to thermal rms emittance. An exact analytical solution of the rms envelope equation is presented, describing both the laminar regime and the transition to the thermal regime: the impact of this new beam equilibrium on the design of high intensity Linacs is discussed.

1 THE INVARIANT ENVELOPE SOLUTION OF RMS ENVELOPE EQUATION IN LAMINAR FLOW

The rms envelope equation for the rms sizes ($\sigma = \sigma_x = \sigma_y$) of a bunched round beam of charged particles under smooth approximation reads[1]:

$$\sigma'' + \frac{p'}{p}\sigma' + \frac{p'^2}{p^2}\Omega^2\sigma = \frac{(I/2I_0)}{p^3\sigma} + \frac{\varepsilon_n^2}{p^2\sigma^3} \quad (1)$$

where $p = \beta\gamma$ is the normalized beam momentum, $p' = \gamma'/\beta = eE_{acc}/mc^2\beta$ the momentum gain rate (assuming a constant momentum rate $p' = \gamma_i'/\beta_i$, which is obtained if $E_{acc} = (mc^2/e)\beta/p'$, we obtain $p = p_0 + p'z$) and I is the peak current in the bunch ($I_0 = ec/r_c$). The rms normalized emittance is ε_n, given by $\varepsilon_n = \sqrt{\langle x^2\rangle\langle p_x^2\rangle - \langle xp_x\rangle^2}$ ($p_x = x'p$), where the averages $\langle\ \rangle$ are performed over the (x, p_x) phase space.

The focusing gradient is given by $K = \Omega^2 p'^2/p^2$, where the normalized focusing frequency Ω, for an ideally synchronous Linac, comes out to be:

$$\Omega = \frac{1}{\cos\varphi_0}\sqrt{\frac{\eta}{8} + \left(\frac{\beta cB}{2E_{acc}}\right)^2 - \frac{\beta^2\cos\varphi_0}{4\gamma^2} + \frac{\beta\sin\varphi_0}{2\alpha}} \quad (2)$$

where B is the field amplitude of focusing solenoids, E_{acc} the accelerating gradient, φ_0 the accelerating phase and $\alpha = eE_{acc}/mc^2k$ the normalized vector potential amplitude of the RF field in the Linac accelerating cavities (having a RF wave number $k = 2\pi/\lambda_{RF}$). The factor $\eta/8$ represents the ponderomotive RF focusing effect[1], η being close to 1 for standing wave cavities and almost vanishing for travelling wave ones (depending on the RF field spatial harmonics). The third and fourth terms under square root in eq.2 represent, respectively, a second order defocusing gradient due to the particle speed change through the cavity and a first order (Panofski-Wenzel) RF defocusing kick imparted to non relativistic particles crossing the cavity [2] (note that φ_0 is usually negative to ensure phase stability).

A similar expression for Ω can be found for a generic quadrupole lattice, as extensively discussed elsewhere[2], in which case the envelope to be considered is the secular one, averaged over the cell to cell oscillations.

The normalized focusing frequency Ω is a constant whenever the solenoid field B is varied along the Linac in such a way to correct for the variation of β, γ and φ_0 in eq.2 so to compensate the first order defocusing RF kicks: in this case the focusing is purely of second order on the secular envelope (*i.e.* scaling like the square of the accelerating gradient and like the inverse square of the momentum). Taking as an example a Linac operated at 500 MHz, with $E_{acc} = 5$ MV/m at injection (100 MeV, $\beta_i = 0.4$, *i.e.* $\alpha = 5\cdot 10^{-4}$) and $E_{acc} = 10$ MV/m at 1 GeV, with $\varphi_0 = -20^o$, the maximum solenoid field, which occurs at injection where the first order defocusing effects from the RF cavities are stronger, comes out to be $B = 0.08\sqrt{93 + 0.9\Omega^2}$, *i.e.* 0.8 T at $\Omega^2 = 8$.

In order to find an equilibrium solution of eq.1 under laminar flow (*i.e.* for a vanishing emittance term in eq.1), we assume that the beam peak current in the bunch is constant: the conditions on the longitudinal dynamics imposed by such a constraint are extensively discussed in ref.2, which reports the prescribed change with p of the accelerating phase φ_0. An exact particular solution, under these conditions $\Omega = const$ $I = const$ and $\varepsilon_n = 0$, is:

$$\hat{\sigma} = \frac{1}{p'}\sqrt{(I/2I_0)/p(1/4 + \Omega^2)} \quad (3)$$

which is an approximate solution of eq.1 whenever the laminarity parameter $\rho = \left\{(I/2I_0)/\varepsilon_n pp'\sqrt{1/4 + \Omega^2}\right\}^2$ is much larger than 1. $\hat{\sigma}$ has been named *invariant envelope* in the field of electron photoinjectors because it's the equilibrium mode for the beam which performs emittance correction. The merit of this exact solution is to treat non perturbatively the effect of acceleration. Indeed, by rewriting the normalized focusing frequency as $\Omega^2 = \eta/8 + (\Omega_L/cp')^2$ (assuming $\cos\varphi_0 = 1$ for simplicity of notation, Ω_L is the Larmor frequency in the solenoid field) the invariant envelope reads $\hat{\sigma} = \sqrt{(I/2I_0)\big/p\left[p'^2(2+\eta)/8 + \Omega_L^2/c^2\right]}$. This clearly shows that equilibrium is possible even without focusing, *i.e.* for $\Omega_L = 0$, because of the focusing due to acceleration: this kind of focusing is completely neglected in usual analysis based on the adiabatic damping approach (see for instance Reiser[3] for an extensive analysis).

On the other hand, the drawback of the invariant envelope description is the lacking of capability to describe the transition from the laminar regime ($\rho >> 1$) into the thermal regime ($\rho \leq 1$), which typically occurs whenever a beam is accelerated (see the scaling of the laminarity parameter as $1/p^2$) from the injector up to the Linac exit. For a proton beam carrying 5 A peak current in the bunch, accelerated in a Linac at 5 MV/m accelerating gradient with a normalized emittance of 1 mm·mrad, the transition ($\rho = 1$) occurs at 0.6 GeV.

2 A QUASI-SOLUTION OF THE RMS ENVELOPE EQUATION TO DESCRIBE THE TRANSITION FROM LAMINAR TO THERMAL FLOW

In order to join the merits of two different descriptions, the one based on the invariant envelope concept, the other based on the tune depression formalism and adiabatic damping (*i.e.* perturbative acceleration), we tried to incorporate the focusing due to acceleration into Reiser's formalism. The equilibrium beam predicted by such description is

$$\sigma_R = \sqrt{\frac{\varepsilon_n}{p\sqrt{K}}}\left\{\frac{1}{2}\frac{(I/2I_0)}{\varepsilon_n p^2\sqrt{K}} + \sqrt{1+\left[\frac{1}{2}\frac{(I/2I_0)}{\varepsilon_n p^2\sqrt{K}}\right]^2}\right\}^{1/2}$$

which reduces to a pure thermal flow $\sigma_{th} = \sqrt{\frac{\varepsilon_n}{p\sqrt{K}}} = \sqrt{\varepsilon\beta^*}$ for $I = 0$ (β^* is the betatron length) and to Brillouin flow $\sigma_{th} = \sqrt{\frac{(I/2I_0)}{p^3 K}}$ for space charge dominated regime ($\varepsilon_n = 0$).

The critical parameter in this description is defined as tune depression $\Delta_K \equiv 1\Big/\left[\nu/2 + \sqrt{1+(\nu/2)^2}\right]$, where $\nu = \frac{(I/2I_0)}{\varepsilon_n p^2\sqrt{K}}$. Δ_K ranges from 0 (Brillouin flow, purely laminar) up to 1 (thermal flow, negligible space charge effects) and basically gives the depression of the applied focusing gradient on the single particle betatron motion due to collective space charge forces.

We notice that Reiser's expression for the matched beam gives the right $1/\sqrt{p}$ behavior (adiabatic damping) with a wrong mix of acceleration focusing and gradient focusing: indeed, it can be rewritten as

$$\bar{\sigma} = \sqrt{\frac{\varepsilon_n}{p\sqrt{K}}\left\{\sqrt{\frac{\rho}{4}\sqrt{\frac{\left(\frac{1}{4}+\Omega^2\right)}{Kp^2/p'^2}}} + \sqrt{1+\frac{\rho}{4}\sqrt{\frac{\left(\frac{1}{4}+\Omega^2\right)}{Kp^2/p'^2}}}\right\}}$$

which becomes, for $K = (\Omega\, p'/p)^2$

$$\bar{\sigma} = \sqrt{\frac{\varepsilon_n}{p\sqrt{K}}\left\{\sqrt{\frac{\rho}{4}\sqrt{\frac{\left(\frac{1}{4}+\Omega^2\right)}{\Omega^2}}} + \sqrt{1+\frac{\rho}{4}\sqrt{\frac{\left(\frac{1}{4}+\Omega^2\right)}{\Omega^2}}}\right\}}$$

clearly revealing the mismatch in describing the *effective focusing from acceleration* (the factor 1/4 is missing because of the adiabatic perturbative treatment of acceleration).

Thus, we make an ansatz by just setting the correct focusing term in Reiser's formula, *i.e.*

$$\bar{\sigma} \equiv \sqrt{\frac{\varepsilon_n}{p'\sqrt{1/4+\Omega^2}}\left[\frac{1}{2}\sqrt{\rho} + \sqrt{1+\frac{\rho}{4}}\right]} \qquad (4)$$

or

$$\bar{\sigma} = \sqrt{\varepsilon_n\left[\frac{1}{2}\sqrt{\rho} + \sqrt{1+\frac{\rho}{4}}\right]\Big/\sqrt{p'^2(2+\eta)\ 8 + \Omega_L^2/c^2}}$$

This expression for $\bar{\sigma}$ will be assumed in the following to be a complete quasi-solution of eq.1, giving an exact non-perturbative description of acceleration. As shown later on, this expression is able to describe the transition from the laminar space charge dominated (tune depression close to zero) flow into the thermal emittance dominated regime. Indeed, the two asymptotical behaviors of eq.4 are

$$\bar{\sigma} \xrightarrow[\rho\to\infty]{} \hat{\sigma} = \frac{2}{p'}\sqrt{\frac{(I/2I_0)}{p(1/4+\Omega^2)}}$$

and

$$\bar{\sigma} \xrightarrow[\rho\to 0]{} \sqrt{\frac{\varepsilon_n}{p\sqrt{K_{eff}}}} = \sqrt{\frac{\varepsilon_n}{\sqrt{p'^2(2+\eta)/8 + \Omega_L^2/c^2}}}$$

In absence of acceleration ($p' = 0$) $\bar{\sigma}$ must be replaced by a Brillouin flow at $\rho >> 1$. Indeed the two flows match automatically one into each other, as shown in ref.1.

In order to check the validity of (4) as a solution of eq.1, first we transform eq.1 into a dimensionless space defined by $\tau \equiv p'\sigma\sqrt{p_0/(I/2I_0)}$ (a normalized beam rms spot size) and $y \equiv \ln(p/p_0)$ (a normalized momentum gain factor (p_0 is the initial beam momentum at injection), obtaining

$$\frac{d^2\tau}{dy^2} + \Omega^2\tau = \frac{e^{-y}}{\tau} + \frac{1}{\rho_0\left(1/4+\Omega^2\right)}\frac{1}{\tau^3} \qquad (5)$$

Here $\rho_0 = \rho(p = p_0)$: the complete solution $\bar{\sigma}$ in physical space transforms into

$$\bar{\tau} \equiv \sqrt{\left(e^{-y}/2 + \sqrt{1/\rho_0 + e^{-2y}/4}\right)\Big/\left(1/4+\Omega^2\right)}$$

clearly displaying the advantage of the dimensionless space: just two free parameters are left, *i.e.* Ω and the initial value ρ_0 at injection of the laminarity parameter, instead of the 4 original ones, *i.e.* Ω (the external focusing) , p' (the accelerating gradient), I (the beam current) and ε_n (the beam emittance). It should also be noted that solution $\bar{\tau}$ is an equilibrium mode for the beam, since it is stable against weak oscillations due to initial mismatches, as proved elsewhere [2].

A comparison between the numerical solution of eq.5 and the previously discussed analytical solutions (*i.e.* the complete quasi-solution $\bar{\tau}$, the pure laminar solution $\hat{\tau} = e^{-y/2}\big/\sqrt{1/4+\Omega^2}$, which represents the invariant envelope transformed into the d-less space, and Reiser's d-less solution τ_R , transformed of σ_R) is shown in fig.1.

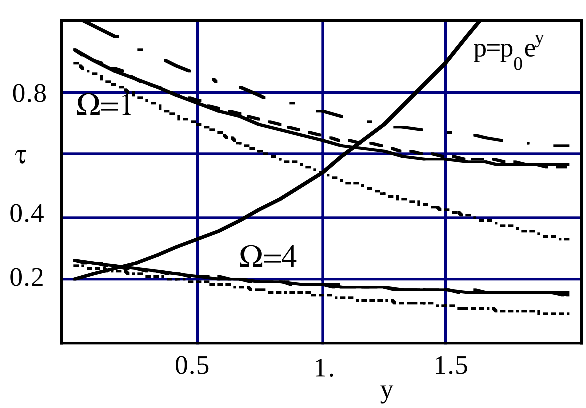

Figure 1: comparison between numerical integration of eq.1 (decreasing solid lines) and analytical solutions for $\Omega = 1$ (upper curves) and $\Omega = 4$ (lower curves). The initial laminarity parameter is $\rho_0 = 10$.

It is clearly shown the excellent agreement between $\bar{\tau}$ (dashed line, which is overlapped to the solid line for the case $\Omega = 4$) and the numerical solution, while the pure laminar solution $\hat{\tau}$ (dotted line) is shown to go unphysically down to zero. On the other hand, Reiser's d-less τ_R (dotted dashed line) clearly displays a lacking of focusing. The proton beam energy ranges in this case from 20 MeV up to about 1 GeV, while he laminarity parameter goes from 10 down to 0.1. Other checks for relativistic electron beams at much higher peak currents are reported elsewhere[4], showing again a very good agreement and the general validity of this model.

It is interesting to compute the tune depression Δ_K corresponding to the solution $\bar{\tau}$. This comes out to be

$$\Delta_K = 1\Big/\left(\sqrt{\rho/4} + \sqrt{1+\rho/4}\right)$$

which becomes, for small value of ρ, *i.e.* across the transition, $\Delta_K \xrightarrow[\rho\to 0]{} 1 - \sqrt{\rho}/2$.

Another interesting quantity is the total phase advance of betatron oscillations along the Linac, defined as

$$\Delta\psi \equiv \int_{z_0}^{z_f} dz/\beta = \int_{p_0}^{p_f} \varepsilon_n dz\Big/\left(p\bar{\sigma}^2 p'\right) \quad , \ i.e.$$

$$\Delta\psi = \frac{1}{4}\sqrt{1+4\Omega^2}\left\{\begin{matrix}\sqrt{\rho_f} - \sqrt{4\sqrt{\rho_f}+\rho_f^2} + \\ 2\ln\left[2/\sqrt{\rho_f} + \sqrt{4/\sqrt{\rho_f}+1}\right]\end{matrix}\right\}$$

where ρ_f is the final value of the laminarity parameter, usually small if the transition into the thermal regime has been completed. If $\rho_f << 1$ we have

$$\Delta\psi \xrightarrow[\rho_f\to 0]{} \sqrt{1/4+\Omega^2}\left\{\ln 2 - (1/4)\ln\rho_f\right\}$$

which gives $\Delta\psi = 2\pi$ for $\Omega = 5$ and $\rho_f = 0.1$.

This quite small value for $\Delta\psi$ can be an attractive mode of operation to avoid the excitation of instabilities in the single particle (betatron) motion, usually driven by collective (space charge) effects giving envelope oscillations starting from mismatches. Since the single particle motion will accomplish just one betatron oscillation in this case, there should be not enough time to drive such an instability, which leads to beam halos.

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