STOCHASTIC CONTROL OF BEAM DYNAMICS

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Abstract

The methods of stochastic control theory are proposed in the context of charged-particle beam dynamics. The stochastic dynamics that is introduced here is invariant for time reversal and can be easily recast in the form of a Schrödinger-like equation where Planck's constant is replaced by the beam emittance. It changes a bilinear control problem for Schrödinger equation in a linear control problem, then resulting more adequate to our aim. This point of view seems to be in agreement with accelerators physics *modus operandi*.

1 INTRODUCTION

The macroscopic state of a particle bunch in an accelerating machine is essentially the result of the collective interaction of the particle among them as well as with the surroundings. However, this macroscopic dynamics involves both several coherent and incoherent microscopic processes whose nature is deterministic or stochastic. The sum of all these processes determines the above macroscopic state whose nature is essentially classical. For example, coherent oscillations of the beam density that are manifested through some mechanism of local correlation and loss of statistical independence may be described by taking into account all the interactions as a whole. Within the context of the conventional descriptions of the beam dynamics, it must be recognized that the study of statistical effects on the dynamics of electron (positron) colliding beams with Fokker–Planck equation for the beam density has received a great deal of attention in literature, stimulating very much the description of the noise sources and dissipation in particle accelerators by standard classical probabilistic techniques [1],[2]. Nevertheless, approaches alternative to the conventional ones should be mentioned for their natural applications to the descriptions of the interaction betwen the beam as a whole and the surroundings. In particular, three approaches are based on a quantum-like formalism which takes into account the diffusion among the beam particles. One of these is known as Thermal Wave Model (TWM) [3] which assumes that the beam dynamics as whole is governed by a Schrödinger-like equation whose diffraction-like term describes the thermal spreading among the electronic rays (diffusion). Another approach is based on a stochastic quantization *ala Nelson* of the beam dynamics in a thermal bath with the environment [4]. Finally, a more recent approach, is based on the simulation of semiclassical corrections to classical dynamics by suitable classical stochastic fluctuations with a suitably defined random kinematics by replacing the classical deterministic trajectories [5]. Additionally, it is worth mentioning that recent experiments on confined classical systems with special phase–space boundary conditions seem to be well described by a quantum–like formalism (Schrödinger-like equation) [6]. In this paper, we use the stochastic formalisms to introduce, as a novel concept, stochastic control theory in beam dynamics. This is done by giving the description of the stability regime for the beam, when thermal dissipative effects are balanced on average by the RF energy pumping, and the overall dynamics is conservative and time–reversal invariant in mean. To this end, we observe that, according to the stochastic formalism, the diffusion process describes the effective motion at the *mesoscopic level* (interplay of thermal equilibrium, classical mechanical stability, and fundamental quantum noise) and therefore the diffusion coefficient is set to be the semiclassical unit of emittance provided by qualitative dimensional analysis. In the next section we model the random kinematics with a particular class of diffusion processes, the Nelson diffusions, that are nondissipative and time–reversal invariant [7]. This allows us to introduce briefly the hydrodynamic equations for the collective stochastic dynamics, and, in turn, to develop control tecniquesfor the beams. In particular, the dynamical equations are derived via variational principle of classical dynamics, with the only crucial difference that the kinematical rules and the dynamical quantities, such as the Action and the Lagrangian, are now random. The stochastic variational principle formally reproduces the equations of the Madelung fluid (hydrodynamic) representation of quantum mechanics with Planck's constant replaced by emittance. In this sense, the present scheme allows us for a quantum–like formulation equivalent to the probabilistic one.

2 STOCHASTIC DYNAMICS

The above quantum-like approaches of beam dynamics are formulated, starting from different physical point of view, but they have the common feature that one can model spatially coherent fluctuations by a random kinematics performed by some collective degree of freedom $q(t)$ representative of the beam. This way, the random kinematics provides an effective description of the space–time variations of the particle beam density $\rho(x,t)$ as it coincides with the probability density of the diffusion process performed by $q(t)$. Then, in suitable units, the basic stochastic kinematical relation is the Ito's stochastic differential equation [7] which, by replacing t with the time-like coordinate $s \equiv ct$ (c being the light speed) and $\hbar$ with the beam emittance ϵ, becomes:

$$dq(s) = v_+(q,s)ds + \epsilon^{1/2}dw \,, \qquad (1)$$

where v_+ is the deterministic drift. Note that the beam emittance plays the role of diffusion coefficient, and dw is the time increment of the standard δ–correlated Wiener noise. We remark that Eq. (1) is equivalent to Fokker-Planck equation.

We are concerning here with the stability regime of the bunch oscillations that can be carried out in a cyrcular accelerator. Thus, in this conditions, the bunch can be considered in a quasi-stationary state, during which the energy lost by dissipation is regained in the RF cavities. In such a quasi-stationary regime, the bunch dynamics is, on average, invariant for time–reversal. We can therefore define a classical *effective* Lagrangian $L(q, \dot{q})$ of the system, where the classical deterministic kinematics is replaced by the random diffusive kinematics (1). The equations for the dynamics can then be obtained from the classical Lagrangian by simply modifying variational principles of classical mechanics into stochastic variational principles.

In the present quantum-like context the analysis is quite similar to the stochastic one [7], yielding again two coupled nonlinear hydrodynamic equations, however, with the emittance replacing Planck's constant in the diffusion coefficient, the real space bunch density replacing the quantum mechanical probability density, and the bunch center velocity replacing the quantum mechanical probability current. Given the stochastic differential equation (1) for the diffusion process $q(s)$ in $3D$-space, in strict analogy with the classical action in the deterministic case, we introduce the following *mean classical action*$[as\Delta s \to 0^+]$:

$$A(s_0, s_1; q) = \int_{s_0}^{s_1} E\left[\frac{1}{2}\left(\frac{\Delta q}{\Delta s}\right)^2 - V(q)\right] ds \,, \qquad (2)$$

where $E(.)$ denotes the conditional expectation with respect to the probability density ρ, and V is the external potential. Note that the *mean classical action* (2) is suitable for the sample paths of a diffusion process that are non differentiable. Consequently, one has the following stochastic variational principle [7]: *under smooth variations of the density $\delta\rho$, and of the current velocity δv, with vanishing boundary conditions at the initial and final times, the Action (2) is stationary, $\delta A = 0$, if and only if the current velocity v (first-order moment of the density ρ) is the gradient of some scalar field $S(x, s)$ (the phase)*:

$$v = \nabla S \,. \qquad (3)$$

Within the above conditions, the two coupled nonlinear Lagrangian equations of motion for the density ρ and for the current velocity v (or alternatively for the phase S) are the following Hamilton–Jacobi–Madelung (HJM) equation:

$$\partial_s S + \frac{v^2}{2} - \frac{\epsilon^2}{2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} + V(x) = 0 \,, \qquad (4)$$

and the continuity equation:

$$\partial_s \rho = -\nabla[\rho v]. \qquad (5)$$

By solving equations (4) and (5) the state of the bunch is completely determined. Note that, by introducing the wave function in the eikonal representation $\Psi(q, s) = \sqrt{\rho}\exp\frac{i}{\epsilon}S$, the above equations are formally equivalent to the Schrödinger-like equation obtained in the three above different approaches [3],[4],[5].

The observable structure is quite clear. The expectations $E(v)$ (first moment of ρ) of the three components of the current velocity v are the average velocities of the bunch center oscillations along the longitudinal and transverse directions. The expectations $E(q)$ (first moments of ρ, as well) of the three components of the process $q(s)$ give the average coordinate of the bunch center. The second moments $\Delta q \equiv \sqrt{E\left((q - E(q))^2\right)}$ of $q(s)$ (r.m.s. of the beam density) allow us to determine the dispersion (spreading) of the bunch. In the harmonic case, these are all the moments that are needed, and we have coherent state solutions. In the anharmonic case the coupled equations of dynamics may be used to achieve a controlled coherence: given a desired state (ρ, v) the equations of motion (4) and (5) can be solved for the external controlling potential $V(x, s)$ that realizes the desired state.

3 CONTROLLED BUNCHES

In order to have a controlled bunch motion, we must control, first of all, the motion of its center. Moreover, we want that the form of bunch does not changes or changes in a controlled way. To this end, we look at the following Ehrenfest's equations

$$\frac{d}{ds}E(q) = E(v) \qquad (6)$$

and

$$\frac{d^2}{ds^2}E(q) = -E(\nabla\Phi). \qquad (7)$$

It is immediately seen, (see Eq.(7)), that all the moments of ρ are involved through the mean values. It is possible, however, to write a set of recursive equations that rules the evolution of all moments. The equation to consider, in general, is that for positional entropy

$$\frac{d}{dt}E(\log\rho) = -E(\nabla v). \qquad (8)$$

Now we illustrate the scheme to construct a simple controlled packets . The idea is the following. If we select a current velocity, we choose, in fact, the caracteristics of the motion of the center of the packet. Moreover, a choice of current velocity selects a class of solutions of continuity (Fokker-Planck) equation. The HJM equations become, in this scheme, a constraint to retain time-reversal invariance, giving us the controlling device.

Let us construct a class of controlled bunches as an example. We need some initial condition ρ_0 for probability density, which satisfies a stationary Schrödinger-like equation with Φ_0 as external potential. By taking the current velocity of the following form [8]

$$v = E(v) + \frac{x - E(q)}{\Delta q}\frac{d\Delta q}{ds}, \tag{9}$$

and inserting the current velocity in continuity equation (7) we solve in a very simple way, obtaining:

$$\rho(\xi) = \int \delta(y - \xi)\rho_0(y)dy \ , \ \xi = \frac{x - E(q)}{\Delta q}. \tag{10}$$

Now, it is not difficult to see that

$$\frac{dE(v)}{ds}x + \frac{1}{2}\xi^2\frac{d^2\Delta q}{ds^2} - \Phi_0(\xi) + L(s) = -\Phi, \tag{11}$$

where Φ_0 is the external potential associated with the solution $\sqrt{\rho_0}$ of the stationary Schrödinger-like equation, Φ is the state dependent control device. By combining (9) (8), we obtain:

$$E(\log \rho(x, s)) = -\log \Delta q. \tag{12}$$

Consequently, the whole positional entropy comes from the dispersion and this means that the set of recursive equations is closed. We can write an equation for Δq, and all the others depend from this last one. The equation is:

$$\frac{d^2\Delta q}{ds^2} = \frac{a}{\Delta q^3} - E(\xi\nabla\Phi) \tag{13}$$

where a is a parameter which depends on the bunch emittance. The Ehrenfest's equation becomes for these states

$$\frac{d^2}{ds^2}E(q) = -\nabla\Phi|_{E(q)}, \tag{14}$$

the *classical* one. It is, also, significant to write Ito's equation for this class of stochastic processes:

$$dq(s) = E(v)ds + \nu dw. \tag{15}$$

They are associated, as Glauber states in quantum mechanics, to Wiener processes with a drift that is solution of the classical (Eherenfest) equation (14).

4 CONCLUSIONS AND REMARKS

In this paper, a method of control theory has been proposed on the basis of recent stochastic approach to particle beam dynamics given in a quantum-like context. This method seems to be in agreement with accelerators physics *modus operandi*. In control theory, in fact, one has the configurational variables of a physical system, then one chooses a velocity field and with a suited strategy one forces the system to have a well defined evolution. The evolution has a cost and the minimitation of the cost is the *premium* for the system controller if he has adopted the right strategy. The strategy is given by well suited laboratory devices (external electromagnetic fields, the characteristics of the accelerators, etc).The fluctuative approach to quantum-like description, as long as it brings to hydrodynamic equations, transforms the difficult problem of control theory for Schroedinger equation (bilinear control) to a linear control for a couple of non-linear equations; it can then be more appropriate to our aim.

5 REFERENCES

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