ESTIMATION OF VERTICAL DISPERSION AND BETATRON COUPLING FOR SOLEIL

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Abstract

Analytical expressions are derived to calculate statistically the vertical dispersion and the induced vertical emittance in the presence of imperfections with and without closed orbit correction. In addition a statistical study is performed to estimate the betatron coupling contribution to the vertical beam size.

1 SPURIOUS VERTICAL DISPERSION

Starting from the classical transverse equations of motion for a charged particle along the s axis, we develop up to second order the magnetic field including dipolar and quadrupolar imperfections :

$$\frac{B_x}{B\rho} = \left[Kz + 2Sxz\right] + \left(N_c + N_{\Delta z} + N_{\Delta\Phi}\right)x + \left(K_{\Delta x} + \Delta K\right)z + \Delta_x$$

$$\frac{B_z}{B\rho} = \left[\frac{1}{\rho} + Kx + S\left(x^2 - z^2\right)\right] - \left(N_c + N_{\Delta z} + N_{\Delta\Phi}\right)z + \left(K_{\Delta x} + \Delta K\right)x + \Delta_z$$

with Bρ, K, S and ρ are respectively the magnetic rigidity, quadrupole and sextupole strengths and the radius of curvature.
$(B\rho)\Delta_x = \Delta B_x(s)$ and $(B\rho)\Delta_z = \Delta B_z(s)$: horizontal and vertical dipolar fields (including dipole correctors), $(B\rho)\Delta K$: gradient error, $K_{\Delta x} = 2S\Delta x$: gradient error induced by horizontal sextupole misalignment, $N_{\Delta z} = 2S\Delta z$: skew gradient error induced by vertical sextupole misalignment, $N_{\Delta\Phi} = 2K\Delta\Phi$: skew gradient error induced by tilted quadrupoles, N_c : skew quadrupole correctors.

The motion can be separated into tree parts : a closed orbit (x_{of},z_{of}), a chromatic closed orbit ($\delta \neq 0$) and the betatron motion. Decomposing again the chromatic closed orbit in the sum of the natural dispersion (D_x, D_z) and the residual dispersion (ΔD_x, ΔD_z) leads to the linear equations for vertical C.O. and for spurious vertical dispersion [1]:

$$z_i^{of} = \sum_j C_{ij}^z \Delta_{zj} \qquad with \qquad C_{ij}^z = c_{ij}^z + \sum_{cor} c_{ic}^z d_{jc}$$

$$\Delta D_i^z = \sum_{quad} c_{iq}^z \left[\left(2KLD_x\right)_q \Delta\Phi_q - \left(KL\right)_q \sum_j C_{qj}^z \Delta_{zj}\right] + \sum_{sext} c_{is}^z \left(2SLD_x\right)_s \left[\Delta z_s + \sum_j C_{sj}^z \Delta_{zj}\right] - \sum_j C_{ij}^z \Delta_{zj} + \sum_n \left(N_c LD_x\right)_n$$

with c_{ij} the vertical response matrix from j to i. We generalize this response matrix C_{ij} including the sum over the correctors : each element d_{jc} represent the value of the corrector c induced by the default j.
In the thin lens approximation with L the length of the element, the terms are : the sum over the quadrupoles of the kick induced by z_{of} and by the tilt, the sum over the sextupole of the kick induced by z_{of} and by the misalignment, the sum over all dipolar errors and the sum over the skew quadrupolar correctors.
If we assume that all the imperfections are distributed with centered normal law and are independent, we are able to derive the r.m.s deviation of the closed orbit and the residual dispersion :

$$\left\langle z_i^{of\,2}\right\rangle = \sum_j \left(C_{ij}^z\right)^2 \left\langle \Delta_{zj}^2 \right\rangle$$

$$\left\langle \Delta D_i^{z\,2}\right\rangle = \sum_{quad}\left(c_{iq}^z 2KLD_x\right)_q^2 \left\langle \Delta\Phi_q^2\right\rangle + \sum_{sext}\left(c_{is}^z 2SLD_x\right)_s^2 \left\langle \Delta z_s^2 \right\rangle + \sum_j \left(\sum_{sext}\left(2SLD_x\right)_s c_{sj}^z - \sum_{quad}\left(KL\right)_q c_{qj}^z - 1\right)^2 \left(C_{ij}^z\right)^2 \left\langle \Delta_{zj}^2\right\rangle$$

Similar expressions hold for the other moments. The mean vertical emittance induced by the residual vertical dispersion in presence of synchrotron radiation is then given by :

$$\left\langle \varepsilon_z^{\Delta D}\right\rangle = \frac{C_q \gamma^2}{\rho J_z} \frac{1}{2\pi\rho} \int_{dipole} \left\langle \Delta H_z \right\rangle ds$$

$$\left\langle \Delta H_z \right\rangle = \gamma_z \left\langle \Delta D_z^2 \right\rangle + 2\alpha_z \left\langle \Delta D_z \Delta D'_z \right\rangle + \beta \left\langle \Delta D'^2_z \right\rangle$$

2 BETATRON COUPLING

To derive a complete analytical formulation of the beam envelop and emittance induced by betatron coupling between the two transverse plans, we study the eigen vectors of the one turn transport coupled matrix [2]. The coupling is induced by skew quadrupolar field generated by tilted quadrupoles or dipoles, vertical sextupole displacement and vertical residual closed orbit in sextupoles. All these effects are treated in the thin lens approximation. Furthermore, if we assume that the vertical emittance is null without coupling errors or vertical residual dispersion, we need to develop all the derivation up to the second order respect to the defaults.

For a 4×4 matrix we have 4 eigen vectors V_1, V_2, V_3 and V_4 . They are two by two conjugate, their module are periodic and represent the optical functions of the machine. The general solution of motion is a linear combination of the form :

$$\vec{Z}(s) = A_1\vec{V}_1 + A_2\vec{V}_2 + A_3\vec{V}_3 + A_4\vec{V}_4$$

with $|A_1| = |A_2|$ and $|A_3| = |A_4|$ being the two constants of motion. For a given particle $\vec{Z}$, the two independent constants are given by :

$$|A_1|^2 = \frac{\left|\tilde{\vec{V}}_1 S\vec{Z}\right|^2}{\left|\tilde{\vec{V}}_1 S\vec{V}_2\right|^2} \quad \text{and} \quad |A_3|^2 = \frac{\left|\tilde{\vec{V}}_3 S\vec{Z}\right|^2}{\left|\tilde{\vec{V}}_3 S\vec{V}_4\right|^2}$$

where S is the 4×4 permutation matrix. From this relations, and integrating over the two independent phase advances, leads to quadratic moments :

$$\overline{Z^2} = \underbrace{\beta_z\varepsilon_x^0\left[\frac{(\sin\mu_+)^2|B|^2 + (\sin\mu_-)^2|A|^2}{(4\sin\mu_+\sin\mu_-)^2} + \frac{\mathrm{Re}\left[\beta_z e^{i\mu_z}\hat{A}B\right]}{8\sin\mu_+\sin\mu_-}\frac{1}{\beta_z}\right]}_{\text{LOCAL COMPONENT}} + \underbrace{\beta_z|A_3|^2}_{\text{GLOBAL COMP0NENT}}$$

with ε_x^0 the horizontal emittance and with the notation :

$$A(s) = \sum_i N_i\sqrt{\beta_{xi}\beta_{zi}}\ \mathrm{Exp}\big(i(\varphi_{xi}(s) + \varphi_{zi}(s))\big)$$

$$B(s) = \sum_i N_i\sqrt{\beta_{xi}\beta_{zi}}\ \mathrm{Exp}\big(i(\varphi_{xi}(s) - \varphi_{zi}(s))\big)$$

$$\mu_+ = \frac{\mu_x + \mu_z}{2} \qquad \text{and} \qquad \mu_- = \frac{\mu_x - \mu_z}{2}$$

where N_i stand for all the above coupling errors and index i run over the lattice elements. Similar expression hold for the two other vertical moments. The global (constant) quantity $|A_3|^2$ is the sum of 3 distinct parts :

- $|A_3^{\beta}|^2$ betatron coupling,
- $|A_3^{\Delta D}|^2 = \varepsilon_z^{\Delta D}$ residual vertical dispersion,
- $|A_3^{\beta/\Delta D}|^2$ mixed term .

They are all coming from the equilibrium state in presence of synchrotron radiation. The former one come from the fact that some imperfections could induced both vertical dispersion and betatron coupling. Finally, the vertical emittance is given by :

$$\varepsilon_z = \varepsilon_x^0\sqrt{\left[\frac{(\sin\mu_+)^2|B|^2 + (\sin\mu_-)^2|A|^2}{(4\sin\mu_+\sin\mu_-)^2} + \frac{|A_3|^2}{\varepsilon_x^0}\right]^2 - \left[\frac{|AB|}{8\sin\mu_+\sin\mu_-}\right]^2}$$

$$\cong \underbrace{\varepsilon_x^0\frac{(\sin\mu_+)^2|B|^2 + (\sin\mu_-)^2|A|^2}{(4\sin\mu_+\sin\mu_-)^2}}_{\text{LOCAL COMPONENT}} + \underbrace{|A_3|^2}_{\text{GLOBAL COMPONENT}}$$

The vertical emittance is a quadratic quantity which respect to the imperfections composed of 4 distinct contributions : a local term (step function with a jump on each errors) and the 3 global components listed above.

If we assume again that all the imperfections are distributed with centered normal law and are independent, the vertical emittance is distributed like a chi-square function of main value :

$$\langle\varepsilon_z\rangle \cong \underbrace{\varepsilon_x^0\frac{(\sin\mu_+)^2\langle|B|^2\rangle + (\sin\mu_-)^2\langle|A|^2\rangle}{(4\sin\mu_+\sin\mu_-)^2}}_{\text{LOCAL COMPONENT}} + \underbrace{\frac{\langle|A_3|^2\rangle}{\varepsilon_x^0}}_{\text{GLOBAL COMPONENT}}$$

Both functions A and B are developed over all the imperfections in a similar fashion than the residual dispersion.

In addition, we develop a complete analysis of the vertical emittance induced by both vertical spurious dispersion and betatron coupling. This scheme is used with a set of random error, and a statistical treatment was performed. Both method will be compared.

3 APPLICATION TO SOLEIL

In this following section, we apply these previous relations to compute the vertical emittance along the cells of the SOLEIL storage ring [3] using the main imperfections listed in table 1. All the above formulations on the residual dispersion and the betatron coupling have been computed in the code BETA [4].

Imperfections	1 r.m.s
Δz, Δx quadrupole	1.10^{-4} m
Δz, Δx sextupole	1.10^{-4} m
Δz, Δs dipole	5.10^{-4} m
$\Delta B/B$ dipole	1.10^{-3}
$\Delta\varphi_s$ quadrupole	2.10^{-4} rad
$\Delta\varphi_s$ dipole	2.10^{-4} rad

Table 1 : Main imperfections used

We first correct the C.O. using 8 dipolar correctors per cell in a statistical way too. The imperfections are summarized in table 1. In average the vertical coupling has jumps only on sextupoles (correlated errors) and we find a mean coupling of 1.92 % (figure 1). This mean coupling is composed of :

1 mean local betatron coupling component : ~0.71 %
2 mean global betatron coupling component : 0.71 %
3 mean vertical dispersion component : 0.50 %

This analytical value is in good agreement with a complete statistical analysis (1.97 %, see figure 2). In addition we plot, for one standard deviation, the residual vertical dispersion on figure 3. The discrepancy between both statistical and analytical results are mainly due to the different kick modeling induced by C. O. in quadrupoles on the vertical dispersion.

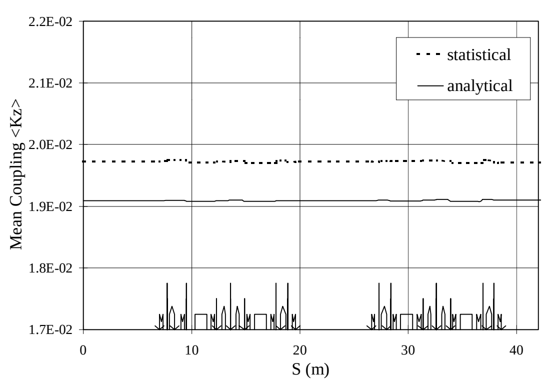

Figure 1 : Mean vertical coupling obtained by a complete analytical derivation along a half period.

In terms of probability, the mean enclose more than 50 % of the cases and two times the mean enclose 90 % of the cases. The coupling distribution is shown in figure 3

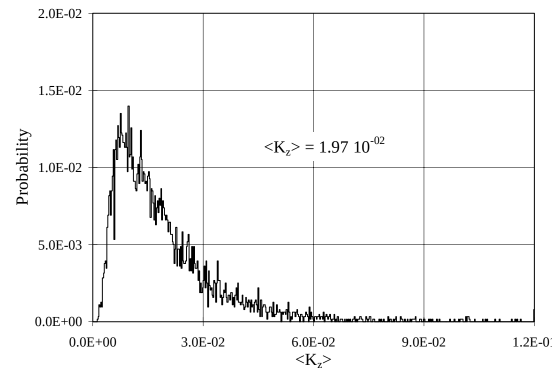

Figure 2 : Distribution function of the vertical coupling obtained by a full statistical computation.

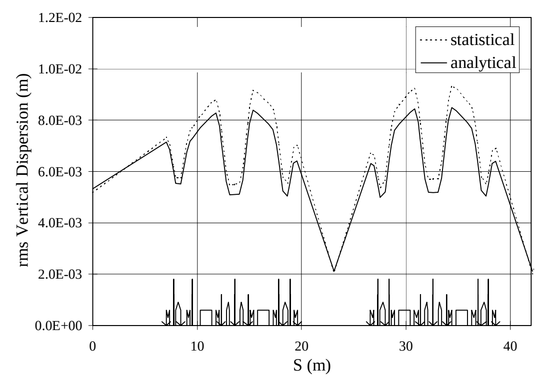

Figure 3 : One standard deviation of the resulting vertical spurious dispersion along a half period.

Contribution to the mean total coupling are :

1. Sextupole vertical displacement : 1.2 %
2. Dipolar imperfections : 0.5 %
3. Tilted quadrupoles : 0.2 %

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