AN IMPEDANCE BOUNDARY CONDITION FOR COMPUTING BEAM COUPLING IMPEDANCES IN COMPLEX-SHAPED PERFORATED BEAM PIPES

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Abstract

An equivalent wall impedance is introduced to describe the electromagnetic boundary conditions at perforated walls, which together with general perturbational formulae can be used to provide accurate analytical estimates of longitudinal and transverse beam coupling impedances in complex-shaped partially perforated pipes.

1 INTRODUCTION

The effect of (a huge number of) pumping holes on beam dynamics and stability, described in terms of coupling impedances, is a fundamental issue and has been carefully investigated, both theoretically [1] and experimentally [2].

With the exception of a few very simple pipe geometris, for which the modal expansion of the electromagnetic field is available, beam coupling impedances can be only computed numerically.

In [3] we developed a general (reciprocity-based, perturbative) analytic approach for computing the (longitudinal and transverse) beam coupling impedances in complex-shaped, heterogeneous pipes, where the local wall properties are described in terms of impedance boundary conditions. In this communication we introduce such a boundary condition for a perforated wall, and discuss a number of relevant generalizations, thus extending the applicability of the aforementioned analytic approach to perforated pipes.

In Sect. 2, we review the perturbative formula for computing the longitudinal beam coupling impedance in a complex-shaped, heterogeneous pipe. In Sect. 3 a perforated wall impedance boundary condition is obtained by solving an appropriate canonical boundary value problem. In Sect. 4 we introduce a modified polarizability, to account for the effect of an external co-axial shield, for an infinite pipe or a large ring. In Sect. 5 a few possible refinements are discussed. Concluding remarks follow under Sect. 6.

2 COUPLING IMPEDANCES IN COMPLEX PIPES

The longitudinal beam coupling impedance $Z_{0,\|}(\omega)$ of a *simple* (e.g., circular and perfectly conducting) pipe can be related to that $Z_{\|}(\omega)$, of *another* pipe differing from the above by some *perturbation* in the boundary geometry and/or electrical properties as shown in [3]:

$$Z_{\|}-Z_{0,\|}=\frac{\epsilon_0}{\beta_0 c Q^2}\left\{\oint_{\partial S}\left[\beta_0 E_n^{(irr.)}(\vec{r},0)+\frac{1}{\beta_0}E_n^{(sol.)}(\vec{r},0)\right]\cdot\right.$$

$$\left.\cdot\frac{Z_w}{Z_0}E_{0n}^{(irr.)*}(\vec{r},0)d\ell-\oint_{\partial S}E_{0z}^{*}(\vec{r},0)E_n^{(irr.)}(\vec{r},0)d\ell\right\}, \quad (1)$$

where ∂S is the cross-section boundary, with (outward) unit normal vector $\hat{u}_n$, and an impedance (Leontóvich) boundary condition

$$\left|(\bar{\bar{I}}-\hat{u}_n\hat{u}_n)\cdot\vec{E}-Z_w\hat{u}_n\times\vec{H}\right|_{\partial S}=0 \qquad (2)$$

describes the (perturbed, local) electrical pipe wall properties. In (1), c, Z_0 and ϵ_0 are the vacuum speed of light, characteristic impedance and permittivity, respectively, β_0 is the relativistic factor, Q is the total beam charge , $\vec{E}_0^{(sol.)}(\vec{r},\vec{r}_0)$, $\vec{E}_0^{(irr.)}(\vec{r},\vec{r}_0)$ are the solenoidal and irrotational parts of the electric field produced at $\vec{r}$ by a line source through $\vec{r}_0$ in the *unperturbed* pipe. The first integral term on the r.h.s of (1), describes the effect of the perturbations in the wall electrical properties, and is nonzero if and only if $Z_w \neq 0$ on ∂S. The second integral term on the r.h.s. of (1), on the other hand, accounts for the effect of the geometrical boundary perturbations, and is accordingly non-zero if and only if the *unperturbed* axial field component E_{0z} is *not* identically zero on ∂S.

3 AN IMPEDANCE B.C. FOR PERFORATED WALLS.

Consider a TM plane wave incident with a nearly-to-grazing angle ($\theta \approx \pi/2$) on a conducting plane at $z=0$ bearing a regular array of holes at $\vec{r}=\vec{r}_{hk}=ha\hat{u}_x+kb\hat{u}_y$, $h,k=-\infty,...,\infty$.

The assumed incident field is similar to the one produced by a relativistic particle beam running parallel to the perfectly conducting wall: the magnetic field is nearly-tangent to the wall, the electric field is nearly normal, and $|E/H|=Z_0$. In this section, following a general procedure discussed e.g. in [4] and applied in [5] we shall deduce from the appropriate reflection coefficient, an equivalent local impedance b. c. to describe the perforated surface.

The field in $z<0$ is the superposition of the incident

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field:

$$\vec{H}^{(i)} = H_0 \hat{u}_y e^{-j\vec{k}^{(i)}\cdot\vec{r}}, \ \ \vec{k}^{(i)} = -\sin\theta \hat{u}_x + \cos\theta \hat{u}_z,$$
$$\vec{E}^{(i)} = Z_0 \vec{H}^{(i)} \times \vec{k}^{(i)} \tag{3}$$

the field reflected from the *unperforated* conducting plane $z = 0$,

$$\vec{H}^{(r)} = H_0 \hat{u}_y e^{-j\vec{k}^{(r)}\cdot\vec{r}}, \ \ \vec{k}^{(r)} = -\sin\theta \hat{u}_x - \cos\theta \hat{u}_z,$$
$$\vec{E}^{(r)} = Z_0 \vec{H}^{(r)} \times \vec{k}^{(i)} \tag{4}$$

and the *scattered* field $\vec{E}^{(s)}$, $\vec{H}^{(s)}$ produced by elementary sources (Bethe's approximation [6]) assumed as radiating in free space (image theorem):

$$\begin{pmatrix} P_z \\ M_y \end{pmatrix} = \sum_{h,k}^{\infty,\infty} 2\delta(\vec{r}-\vec{r}_{hk}) \begin{pmatrix} \alpha_e \epsilon_0 E_z^{(i+r)}(\vec{r}_{hk}) \\ \alpha_m H_y^{(i+r)}(\vec{r}_{hk}), \end{pmatrix}, \tag{5}$$

where α_e, α_m are the (internal) electric and magnetic hole polarizabilities[1].

The scattered field can be readily computed from the vector ($\vec{A}$) and magnetic-Hertz ($\vec{\Pi}$) potentials,

$$\vec{H}^{(s)} = \mu_0^{-1} \nabla \times \left(\vec{A} + \mu_0 \nabla \times \vec{\Pi}\right),$$
$$\vec{E}^{(s)} = j\omega \nabla \times \left(c^{-2} \nabla \times \vec{A} - \mu_0 \vec{\Pi}\right), \tag{6}$$

solving the wave equations:

$$(\nabla^2 + k_0^2) \begin{pmatrix} \vec{A} \\ \vec{\Pi} \end{pmatrix} = -\begin{pmatrix} j\omega\mu_0 \vec{P} \\ \vec{M} \end{pmatrix}. \tag{7}$$

Using the (generalized) Fourier representation:

$$\sum_{h,k}^{-\infty,\infty} \delta(\vec{r}-\vec{r}_{hk}) = \frac{\delta(z)}{ab} \sum_{p,q}^{-\infty,\infty} e^{-j2\pi[(px/a)+(qy/b)]} \tag{8}$$

in (5), the (weak) solutions of (7) can be written as superpositions of plane waves:

$$\begin{pmatrix} \vec{A} \\ \vec{\Pi} \end{pmatrix} = \sum_{p,q}^{-\infty,\infty} \begin{pmatrix} A_{pq}\ \hat{u}_z \\ \Pi_{pq}\ \hat{u}_y \end{pmatrix} e^{j\gamma^{(pq)}z} \cdot$$
$$\cdot e^{-j\{k_0 x \sin\theta + 2\pi[(px/a)+(qy/b)]\}} \tag{9}$$

propagating in the grating-lobe directions, with

$$\begin{pmatrix} A_{pq} \\ \Pi_{pq} \end{pmatrix} = -\frac{2H_0}{ab\,\gamma_{pq}} \begin{pmatrix} k_0 \sin\theta \mu_0 \alpha_e \\ j\alpha_m \end{pmatrix} \tag{10}$$

and[2] :

$$\gamma_{pq} = \sqrt{k_0^2 - (2\pi q/b)^2 + [k_0 \sin\theta + (2\pi p/a)]^2}. \tag{11}$$

[1]The polarizabilities for several hole shapes can be found in [7].

[2]All terms with $(p,q) \neq (0,0)$ in (9) decay exponentially off the $z = 0$ plane, in the limit $\theta \to \pi/2$, as seen from (11).

The $p = q = 0$ term in (9) propagates in the specular direction. The corresponding magnetic field :

$$\vec{H}_{00}^{(s)} = -2jk_0 \frac{\alpha_m + \alpha_e \sin^2\theta}{ab\ \cos\theta} H_0\ e^{-j\vec{k}^{(r)}\cdot\vec{r}}\ \hat{u}_y \tag{12}$$

can be used to compute the reflection coefficient:

$$\Gamma_H = \frac{H_{00}^{(s)} + H^{(r)}}{H^{(i)}} = 1 - 2jk_0 \frac{\alpha_m + \alpha_e \sin^2\theta}{ab\ \cos\theta}. \tag{13}$$

By comparison with the well known formula [3]

$$\Gamma_H = -\frac{Z_+ - Z_-}{Z_+ + Z-} \approx 1 - \frac{2Z_+}{Z_-} \tag{14}$$

which gives the (magnetic) reflection coefficient at the boundary between half spaces with characteristic oblique impedances Z_- (incident wave half-space) and Z_+, letting, in view of (3), $Z_- = Z_0 cos\theta$, we get:

$$Z_+ = -\frac{jk_0 Z_0}{ab}(\alpha_m + \alpha_e \sin^2\theta) \tag{15}$$

which, for near-to-grazing incidence is[4]:

$$Z_+ = -jk_0 Z_0 n_\sigma (\alpha_m + \alpha_e) := Z_w, \tag{16}$$

where $n_\sigma = (ab)^{-1}$ is the hole surface-density. As a conclusion, we can state that for near-to-grazing incidence, a plane perforated conducting surface can be described using an impedance b.c., with wall-impedance given by (16). Moreover, according to the general theory formulated in [8], provided the further condition:

$$|(Z_0/Z_w)\ k_0\ \rho_S| \gg 1 \tag{17}$$

is satisfied, where ρ_s is the (local) smallest radius of curvature of the perforated surface S, then an impedance b.c. with wall-impedance Z_w can still be used for a *non-planar* perfectly conducting perforated surface.

4 BEAM COUPLING IMPEDANCE IN PERFORATED PIPE FROM IMPEDANCE B.C.

Let $n_\sigma = N_\lambda \delta(\ell - \ell_h)$, i.e., assume N_λ (uniformly spaced) holes per unit pipe lenght[5], located at $\ell = \ell_h$, ℓ being the arc-lenght along the pipe cross-section contour. Then, from (1) and (16),

$$Z_{\|} \ = \ -jZ_0 k_0 (\alpha_e + \alpha_m)\ e_n(\ell_h) e_n^*(\ell_h), \tag{18}$$

where $e_n(\ell_h) = (Q/\epsilon_0)^{-1} E_n(\ell_h)$, $E_n(\ell_h)$ being the (normal) electric field produced by an axial beam with total charge Q at the hole's position. Equation (18) reproduces Kurennoy's result valid for this (most) general case, obtained using a different rigorous (modal) approach [9].

[3]The approximate equality holds to lowest order in the ratio Z_+/Z_-, assumed suitably small.

[4]In the near-to-grazing incidence limit $\theta \approx \pi/2$ we still assume that $(ab\cos\theta)^{-1} k_0(\alpha_e + \alpha_m) \ll 1$.

[5]For the very definition of beam impedance to apply, the holes should be (at least piecewise) uniformly-distributed in the longitudinal direction.

5 PERFORATED BEAM PIPE IN A CO-AXIAL SHIELD

For an infinitely long (or equivalently, a large [6] annular) perforated beam pipe surrounded by a co-axial shield (e.g., the LHC cold-bore) eq. (16) still holds, provided the following *modified* polarizabilities are used in place of $\alpha_{e,m}$:

$$\alpha^{(int.)}_{e,m} + F\alpha^{(ext.)}_{e,m}, \qquad (19)$$

where the superfix identifies the *internal* and *external* hole polarizabilities, and

$$F = -\frac{\alpha^{(ext.)}_{e} + \alpha^{(ext.)}_{m}}{\alpha^{(int.)}_{e} + \alpha^{(int.)}_{m} + jn_z^{-1}\delta_S^{*}\frac{\oint_{\partial S_c}|e_c|^2 d\ell}{|e_c(\vec{\rho}_h)|^2}}, \qquad (20)$$

where δ_S is the complex EM penetration depth into the external beam pipe and internal coaxial shield surfaces, ∂S_c is the (complete) contour of the co-axial region cross-section, e_c is the TEM eigenfunction in the co-axial region, $\vec{\rho}_h$ is the transverse hole position, and n_z the longitudinal hole density.

Equation (20) is a generalization of a result by Gluckstern [10], valid for the circular co-axial geometry, and is readily established [11] by computing the field in the co-axial region produced by a longitudinally periodic array of holes at $z = pL$, $p = -\infty, \ldots, \infty$, and $\rho = \rho_h$, and letting

$$E^{(ext.)}(\vec{\rho}_h, pL) = FE^{(int.)}(\vec{\rho}_h, pL), \quad \forall p, \qquad (21)$$

where $E^{(int.)}$, $E^{(ext.)}$ are the pipe and co-axial region fields, and F is *independent* of p, as a consequence of problem's linearity and invariance under the group of longitudinal shifts by multiples of L.

6 THICK-HOLES AND MORE

It is relatively straightforward to include e.g. the effect of i) a non-zero wall thickness; ii) the electromagnetic coupling among the holes . Internal and external electric and magnetic polarizabilities of *thick* holes have been computed by Gluckstern [12]. Coupling among the holes can be accounted by using the *effective* electric polarizabilities $\alpha'_{e,m}$ of a hole belonging, e.g., to an infinite, regular planar hole array:

$$\alpha'_{e,m} \; = \; \alpha_{e,m}/(1 - C_{e,m}\alpha_{e,m}), \qquad (22)$$

where [5]:

$$C_m = -\frac{C_e}{2} = a^{-3}\left[6/5\pi - 8\pi\, K_0(2\pi)\right], \qquad (23)$$

a being the interhole spacing, and K_0 being the Bessel function of the $3rd$ kind[7].

Note that all corrections to the hole polarizabilities should be gauged consistently against the omission [13] of terms of higher order in $k \cdot (hole\ size)$ in the standard no-thickness, no-coupling (Bethe's) formulae for $\alpha_{e,m}$.

[6]Bunch lenght $\ll$ ring circumference.

[7]Equation (23) is obtained in the quasi static ($a \ll bunch\ lenght$) assumption.

7 CONCLUSIONS AND HINTS FOR FUTURE RESEARCH

A simple impedance boundary condition which can be used to compute the beam coupling impedance in partially perforated pipes with complicated geometries in analytic form has been introduced.

Possible hints for future work include the statistical characterization of the beam coupling impedances for (transversely) random-placed holes, the formulation of higher order (variational) coupling impedance perturbation formulas, and the possible use of higher order impedance boundary conditions. Work in these directions is in progress.

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