HIGHER ORDER FORMULA FOR NONLINEAR DISPERSION

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Abstract

We perturbatively formulate the nonlinear dispersion and give the recursion expressions for the higher order terms up to the fourth order. As an example, the nonlinear dispersion function of the storage ring of SPring-8 is numerically calculated. An experimental study of the nonlinear dispersion was also carried out at the storage ring and it was found that the agreement between the theory and the measurement was fairly good up to the second order.

1 INTRODUCTION

Recently, for the stored beam in an electron storage ring, there has been increasing interest in making the bunch length short. One of the important essences for the short bunch is the smallness of the momentum compaction factor. In order to make a momentum compaction factor extremely low, one should control not only the linear part of the dispersion function but also the nonlinear part. Hence the development of analytical expression for the nonlinear dispersion is significant to give a scheme of precisely controlling a momentum compaction factor. Up to the second order of perturbation the nonlinear dispersion was studied by a second-order transfer matrix method [1] and by simplified differential equations of motion [2]. The latter method is completed by J.-P. Delahaye and J. Jäger [3] who gave the formal expression of the nonlinear dispersion up to the second order. We then accomplish the closed form of the recurrent equations for higher order terms of the dispersion function by means of the Hamiltonian formalism and solve the equations perturbatively up to the fourth order. The detail of the present report can be found in [4].

2 FORMULATION

We start with the Hamiltonian for a paricle motion with a momentum deviation δ from the design value

$$\hat{H} = -\left(1+K_x\cdot x\right)\sqrt{\left(1+\delta\right)^2-\hat{p}_x^2}+\frac{1}{2}\left(1+K_x\cdot x\right)^2 +\frac{1}{2}g_0x^2+\frac{1}{6}\lambda_0x^3+\frac{1}{24}\rho_0x^4, \quad (1)$$

where K_x is the curvature of the bending magnet, and g_0, λ_0 and ρ_0 the strength of the quadrupole, sextupole and octupole magnets, respectively. The equation of motion for a particle with off-momentum is then written down as

$$x' = \left(1+K_x\cdot x\right)\frac{\hat{p}_x}{\sqrt{\left(1+\delta\right)^2-\hat{p}_x^2}}, \quad (2)$$

$$\hat{p}_x' = -\left(K_x^2+g_0\right)x-\frac{1}{2}\lambda_0x^2-\frac{1}{6}\rho_0x^3 +K_x\left[\sqrt{\left(1+\delta\right)^2-\hat{p}_x^2}-1\right], \quad (3)$$

whose periodic solution is the dispersion function. Here we assume that there is no vertical bending field, so that we have no vertical dispersion. Expanding the horizontal motion with respect to the momentum deviation δ

$$x = \eta_0\delta+\eta_1\delta^2+\eta_2\delta^3+\eta_3\delta^4+\eta_4\delta^5+\mathcal{O}\left(\delta^6\right), \quad (4)$$

$$\hat{p}_x = \xi_0\delta+\xi_1\delta^2+\xi_2\delta^3+\xi_3\delta^4+\xi_4\delta^5+\mathcal{O}\left(\delta^6\right) \quad (5)$$

and eleminating the momenta ξ_n, we derive the recurrent equations for higher order terms of the nonlinear dispersion function η_n, which have the following general expression:

$$\eta_n''+\left(K_x^2+g_0\right)\eta_n = \Omega_n\left(\eta_0,\cdots,\eta_{n-1}\right). \quad (6)$$

Here the inhomogeneous term $\Omega_n\left(\eta_0,\cdots,\eta_{n-1}\right)$ is derived perturbatively as

$$\begin{aligned}
\Omega_0 &= K_x,\\
\Omega_1 &= g_0\eta_0-\frac{1}{2}\lambda_0\eta_0^2-K_x\left(1-\frac{1}{2}\eta_0'^2\right)\\
&\quad +2K_x^2\eta_0-K_x^3\eta_0^2,\\
\Omega_2 &= g_0\left(\eta_1-\eta_0-\frac{3}{2}\eta_0\eta_0'^2\right)-\lambda_0\left(\eta_1-\frac{1}{2}\eta_0\right)\eta_0\\
&\quad -\frac{1}{6}\rho_0\eta_0^3+K_x\left(1+\eta_1'\eta_0'+\frac{3}{2}\eta_0'^2\right)\\
&\quad +2K_x^2\left(\eta_1-\eta_0-\eta_0\eta_0'^2\right)-K_x^3\left(2\eta_1-\eta_0\right)\eta_0,
\end{aligned}$$

and so on. Since the recurrent equations have the same form as harmonic oscillation with a driving force, the formal solutions of the equations can be easily obtained in terms of the Green function [5]. One then finds that the highest pole of the magnetic field appearing in the explicit expressions of the nonlinear dispersion corresponds to the order of the expansion. For example, the sextupole magnet first appears at the first order, the octupole at the second order, and so on. This fact suggests that in principle we can independently adjust each order term of the nonlinear dispersion by using magnets with suitable multipole field.

Once the nonlinear dispersion are known, we can easily calculate the higher order terms of the momentum compaction factor from the following expression:

$$\alpha\delta = \frac{1}{L}\int_0^L ds\left[\frac{\sqrt{1+\beta_0^2\left(2\delta+\delta^2\right)}}{1+\delta} \times\sqrt{\left(1+K_x\cdot x\right)^2+x'^2}-1\right], \quad (7)$$

where $\beta_0=\sqrt{1-1/\gamma_0^2}$. At the lowest order we have

$$\alpha_0 = \frac{1}{L}\int_0^L ds\left[K_x\left(s\right)\eta_0\left(s\right)-\frac{1}{\gamma_0^2}\right], \quad (8)$$

which is the usual linear momentum compaction factor. The next order term is

$$\alpha_1 = \frac{1}{L}\int_0^L ds\left[K_x(s)\left\{\eta_1(s) - \frac{1}{\gamma_0^2}\eta_0(s)\right\} + \frac{1}{2}\eta_0'^2(s) + \frac{1}{2\gamma_0^2}\left(3 - \frac{1}{\gamma_0^2}\right)\right], \quad (9)$$

and so on. The momentum compaction factor can be regarded as the average of the dispersion function which oscillates over the ring. Hence the error is canceled out on an average, and the momentum compaction factor is hard to be affected by the error. In fact, the numerical calculations show that the fourth order term of the momentum compaction factor is scarcely affected by the distortion of the linear dispersion function while the fourth order of the dispersion changes largely.

3 NUMERICAL STUDY

The numerical integration can be carried out by the transfer matrix method. The driving terms of the equations can be regarded as the curvature of a sector magnet and hence the periodic solutions of the harmonic equations are easily derived from the transfer matrix. Order by order the one turn matrix is constructed from a sequential product of piece wise matrices of magnetic elements for the corresponding order dispersion function η_n. Since the n-th order term of the nonlinear dispersion is generated by the n-th order dipole fields, the n-th order dispersion can be described by the same form as sector bending magnets. For the storage ring of SPring-8, which is the brilliant light source facility with electron beam energy 8 GeV, the nonlinear dispersion functions for the design optics are numerically calculated up to the fourth order. We show each order term of the dispersion function form the zeroth to the third in Figs. 1, 2, 3 and 4. As the actual orbit is not known precisely, we use the design optics to calculate the nonlinear dispersion. To clarify the condition where we can use the design optics for the calculation of the nonlinear dispersion, we calculate it for the cases with some realistic closed orbit distortion (COD). The result implies that (see Fig. 5), if the distortion of the linear dispersion is small, up to the third order there is little difference between the dispersion calculated from the design optics and that of the actual optics with some COD's. On the other hand, one finds that at the fourth order the difference grows larger. It is then concluded that up to the third order we can discuss the nonlinear dispersion by means of our formulation with the design optics under the condition that the leakage of the linear dispersion is made to several percent order compared to the peak value.

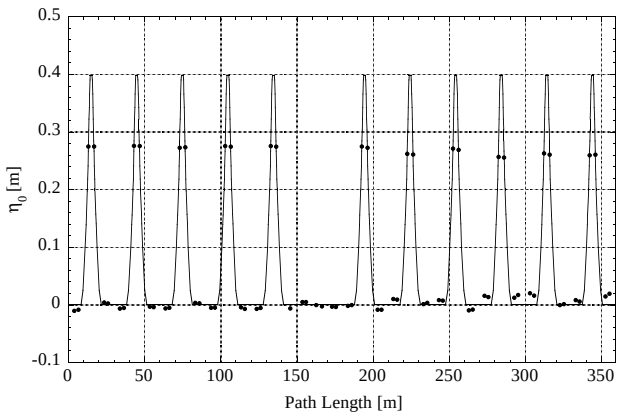

Figure 1: Linear dispersion function in a quarter of the storage ring. The solid line denotes the numerical calculation and the dots the measured values.

4 EXPERIMENTAL STUDY

The measurement of the nonlinear dispersion was carried out at the storage ring, where the COD is corrected less than one hundred μm in r.m.s. value and hence the leakage of linear dispersion is less than several percent of the peak value. The measured values of the dispersion function form the zeroth to the third are expressed by the dots in Figs. 1, 2, 3 and 4, respectively. We found that up to the second order there are no remarkable discrepancy between the theory and the experiment. On the contrary to the numerical calculation, we do not see a good agreement between the theoretical and the measured values at the third order of the dispersion. This can be explained by the narrow available momentum range to fit the dispersion. In the narrow range, the contribution of the fourth power of the momentum deviation to the dispersion, i.e. the third order term becomes comparable to the order of the magnitude of the BPM noise level, several μm.

In order to confirm the significance of the polynomial expansion of the measured dispersion function, we calculated the residual of the polynomial fitting by changing the order of the expansion (see Fig. 6). One finds that up to the second order the precision of the polynomial expansion of the dispersion function is improved as the order of the expansion rises and that at the third order it begins to sat-

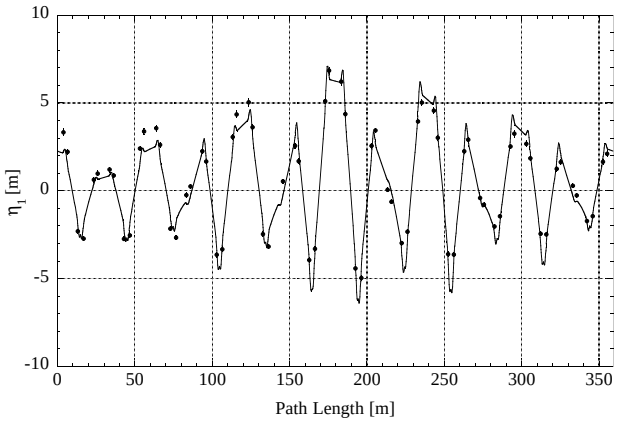

Figure 2: The first order nonlinear dispersion function in a quarter of the storage ring.

urate. This limitation comes from the narrow momentum acceptance, which makes it difficult to fit the higher order terms by the polynomial approximation.

5 DISCUSSIONS

We derived the recurrent expressions for the higher order terms of the nonlinear dispersion function for a ring with a large radius of curvature. The measurement of the nonlinear dispersion at SPring-8 agrees fairly well with the formula up to the second order. One of the reasons of this good agreement is that since the COD is well corrected, the estimation of the nonlinear dispersion by using the design optics approximates to the actual one. On the other hand, the reason of the larger disagreement at the third and higher orders is that the range of the momentum deviation over which one fits the nonlinear dispersion is limited by the momentum acceptance: although the larger momentum deviation is necessary for estimating the higher order terms of the nonlinear dispersion, only the limited region of the momentum is available. In the restricted region the difference of the orbit due to the higher order terms is small, so that it is difficult to derive the higher order terms by means of the polynomial fitting.

6 REFERENCES

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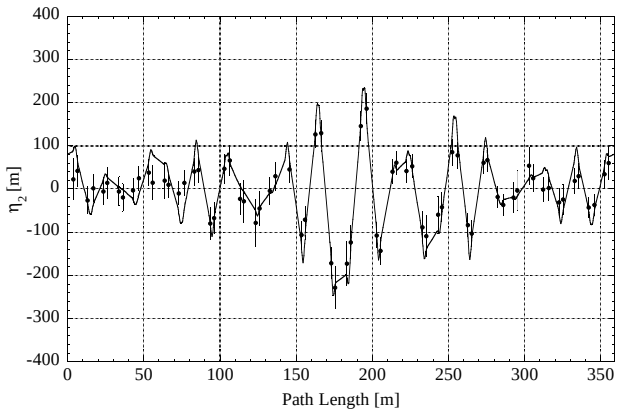

Figure 3: The second order nonlinear dispersion function in a quarter of the storage ring.

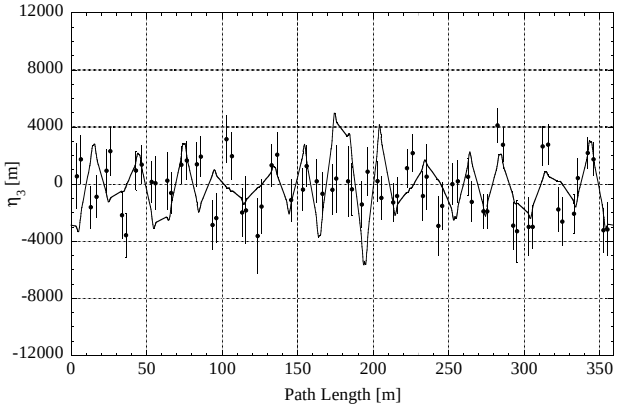

Figure 4: The third order nonlinear dispersion function in a quarter of the storage ring.

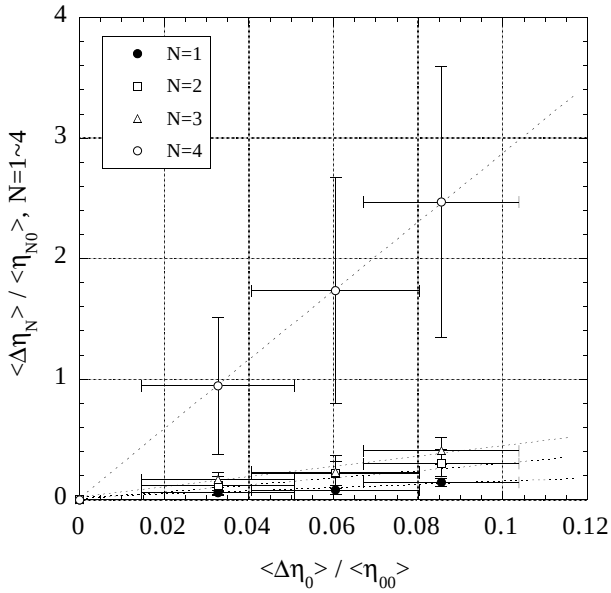

Figure 5: The dependences of the variations of the higher order functions of the nonlinear dispersion on the deviation of the linear dispersion from the design value.

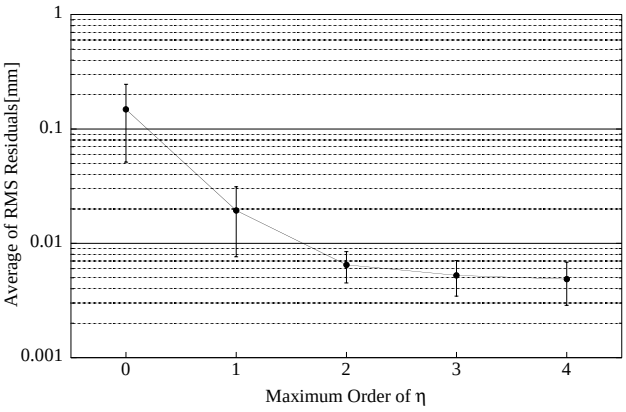

Figure 6: The variation of the average residual of the dispersion function as changing the order of polynomial expansion.