

Symplectic Propagation of the Map, Tangent Map and Tangent Map Derivative through Quadrupole and Combined-Function Dipole Magnets without Truncation

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Abstract

The MAPA (modular accelerator physics analysis) accelerator modeling code¹ symplectically advances the full nonlinear map, tangent map and tangent map derivative through all accelerator elements.² The tangent map and its derivative are nonlinear generalizations of Brown's first- and second-order matrices [1], and they are valid even near the edges of the dynamic aperture, which may be beyond the radius of convergence for a truncated Taylor series. In order to avoid truncation of the map and its derivatives, the Hamiltonian is split into pieces for which the map can be obtained analytically. Yoshida's method [2] is then used to obtain a symplectic approximation to the map, while the tangent map and its derivative are appropriately composed at each step to obtain them with equal accuracy. We discuss our splitting of the quadrupole and combined-function dipole Hamiltonians and show that very few steps are required for typical magnets of a high-energy accelerator.

1 THE MAP & ITS DERIVATIVES

The n -dimensional phase space map describing propagation of a particle through an accelerator element consists of n equations of the form:

$$\bar{p}_i = \bar{p}_i(p_1, p_2, \dots, p_n), \quad \text{Eq. (1)}$$

where the overstrike indicates the final value of a phase space variable, lack thereof indicates an initial value, and i ranges from 1 to n . Ray tracing requires that one accurately calculate the n functions of Eq. (1), either analytically or numerically.

The tangent map is an n by n matrix, with coefficients T_{ij} given by the partial derivatives of the map:

$$T_{ij} = \frac{\partial}{\partial p_j} \bar{p}_i(p_1, p_2, \dots, p_n), \quad \text{Eq. (2)}$$

where i and j both range from 1 to n . If the map is derived from a Hamiltonian, as is the case for accelerators in the absence of synchrotron radiation, then the tangent map is a symplectic matrix. Evaluating the tangent map on the accelerator axis (i.e. setting all p_1 through p_n in Eq. (2) to zero) destroys its symplectic character and yields the familiar linear R-matrix [1], with coefficients independent of the phase space variables.

The tangent map derivative is an n by n by n tensor, with coefficients S_{ijk} given by the second partial derivatives of the map:

$$S_{ijk} = \frac{\partial}{\partial p_k} T_{ij} = \frac{\partial^2}{\partial p_j \partial p_k} \bar{p}_i(p_1, p_2, \dots, p_n), \quad \text{Eq. (3)}$$

where i, j and k all range from 1 to n . Evaluating the tangent map derivative on the accelerator axis (i.e. setting all p_1 through p_n in Eq. (3) to zero) yields the familiar second-order TRANSPORT coefficients [1], within a numerical factor.

2 ALTERNATIVE TO DA APPROACH

Given the map, tangent map and tangent map derivative for a periodic structure, one has all the information required by root-finding algorithms for obtaining any and all fixed points of the map (i.e. closed phase space trajectories), even if these trajectories are far from the accelerator axis. Furthermore, once a closed trajectory of interest has been found, one then has all the information required to calculate and propagate the quantities describing particle motion in the vicinity: Courant-Snyder (also known as Twiss, or for the case of fully-coupled motion, Mais-Ripken) parameters, linear and second-order dispersions, chromaticities, etc.

Of course, one can calculate a truncated form of the map with a DA (differential algebra) package (see e.g. Ref.'s [3] and [4]), and then trivially obtain truncated forms of the tangent map and its derivative. The power of the DA approach, when coupled with normal form methods, [5] is undeniable. However, there is always the danger that the associated multi-dimensional truncated power series does not converge in phase space regions far from the accelerator axis, and such regions can be of interest. Thus, it can be helpful to have an alternative approach when, for example, studying phase space dynamics near the limits of the dynamical aperture.

This is accomplished in the MAPA code by using the Yoshida [2] method, for which the Hamiltonian of a finite-length element is split into two or more exactly solvable pieces, such that the map, tangent map and tangent map derivative can be obtained analytically for each piece. When composing the map in appropriate Yoshida fashion for each step, the tangent map and its derivative can also be composed. Thus, all three quantities are obtained for the full element, and the tangent map is guaranteed to be symplectic to machine accuracy.

¹ Work supported by Tech-X Corporation and by U.S. DOE grant no. DE-FG03-96ER82292.

² See URL <http://www.techxhome.com> for details of the free C++ accelerator modeling class library used by MAPA.

3 FINITE-LENGTH QUADRUPOLE

The Hamiltonian describing particle dynamics in a finite-length idealized quadrupole magnet has the form

$$K = \left(\frac{1 + \alpha \delta_s}{\beta_s} \right) \tilde{P}_\tau + \frac{G_1}{2B_\rho} (x^2 - y^2) - \sqrt{1 + \frac{2}{\beta_s} \tilde{P}_x + \tilde{P}_x^2 - \tilde{P}_y^2 - \tilde{P}_y^2} \quad \text{Eq. (4)}$$

where α is the momentum compaction factor, δ_s is the relative deviation between reference and design momenta, G_1 is the quadrupole gradient in T/m, B_ρ is the magnetic rigidity in T-m, the canonical phase space variables are:

$$p_1 = x \text{ (meters); } p_2 = \tilde{P}_x = \beta_x \gamma / \beta_s \gamma_s; \quad \text{Eq. (5a)}$$

$$p_3 = y \text{ (meters); } p_4 = \tilde{P}_y = \beta_y \gamma / \beta_s \gamma_s; \quad \text{Eq. (5b)}$$

$$p_5 = \delta\tau = c(t_s - t) \text{ (meters); } p_6 = \tilde{P}_\tau = (\gamma - \gamma_s) / \beta_s \gamma_s; \quad \text{Eq. (5c)}$$

and we have followed the MAD-8 conventions [6]. This Hamiltonian must be split into two pieces, in order to obtain the map via the Yoshida method.

The approach taken in the tracking code TEAPOT [7] and some other codes is to split a finite-length quadrupole into N thin quadrupoles, which is equivalent to a drift-kick splitting of the Hamiltonian, where the $(x^2 - y^2)$ term is separated from the other terms. It has long been advocated [8] (and implemented within the SAD code) [9] that a more accurate approach is to split the Hamiltonian into a piece that generates the dynamics of a purely linear quadrupole and a piece that represents nonlinear chromatic effects.

The split used in MAPA is $K = K_{lq} + K_{chro}$, where

$$K_{lq} = \frac{\alpha \delta_s}{\beta_s} \tilde{P}_\tau + \frac{1}{2\beta_s^2 \gamma_s^2} \tilde{P}_\tau^2 + \frac{1}{2} \left(\tilde{P}_x^2 + \frac{G_1}{B_\rho} x^2 \right) + \frac{1}{2} \left(\tilde{P}_y^2 - \frac{G_1}{B_\rho} y^2 \right) \quad \text{Eq. (6)}$$

and

$$K_{chro} = \mathcal{O} \left(\tilde{P}_\tau^3, \tilde{P}_\tau^2 \tilde{P}_x, \tilde{P}_\tau^2 \tilde{P}_y \right) \\ = \left(\frac{\tilde{P}_\tau}{\beta_s} \right) - \frac{1}{2} \left(\frac{\tilde{P}_\tau}{\beta_s \gamma_s} \right)^2 - \frac{1}{2} \left(\tilde{P}_x + \tilde{P}_y \right) - \sqrt{1 + \frac{2\tilde{P}_\tau}{\beta_s} + \tilde{P}_\tau^2 - \tilde{P}_x^2 - \tilde{P}_y^2} \quad \text{Eq. (7)}$$

It is shown in Table 1 that, for a given number of steps, this splitting is far more accurate than a simple drift-kick splitting. A theoretical explanation for this improvement in accuracy has been developed recently [10] in the context of using Zassenhaus formulas to compute the symplectic map for a truncated Hamiltonian -- an approach that has been implemented in the code MaryLie 5.0. [11]

Table 1: The number of integration steps required to achieve a specified relative accuracy in the map for 4 GeV electrons traversing a quadrupole with $L = m$ and $k = 1m^{-2}$

accuracy	1.0e-04	1.0e-06	1.0e-08	1.0e-09
drift-kick	2	10	84	256
lq-chro	1	2	4	10

The Yoshida approach used in MAPA has the advantage that it does not truncate the equations of motion, as is done in the Lie algebra approach used by the MaryLie code. Also, this approach is more accurate (for the same number of steps) than TEAPOT's use of thin quadrupoles. More importantly, the effect of arbitrary high-order multipoles distributed throughout the body of the quadrupole magnet can be correctly included by adding an appropriate third term to the Hamiltonian.

All components of the map, tangent map and tangent map derivative have been calculated for the Hamiltonians of Eq. (6) and Eq. (7). [12] These results have been implemented in the MAPA code and tested by verifying numerically that the tangent map is symplectic and by numerically generating first and second partial derivatives of the map for direct comparison with the analytic results.

4 THE SECTOR BEND

A particle traversing an idealized sector bend is a special case of charged particle motion in a uniform magnetic field. Thus, the map can be obtained analytically. This map is presented in Ref. [13] for the ultra-relativistic limit ($\beta=1$, γ finite) and was implemented in the SAD code [9]. The map was generalized to finite β and implemented in the TOPKARK code. [14]

The idealized sector bend Hamiltonian has the form:

$$K_{sb} = \left(\frac{1 + \alpha \delta_s}{\beta_s} \right) \tilde{P}_\tau + \frac{x}{(1 + \delta_s) \rho_0} \left(1 + \frac{x}{2\rho_0} \right) - \left(1 + \frac{x}{\rho_0} \right) \sqrt{1 + \frac{2}{\beta_s} \tilde{P}_x + \tilde{P}_x^2 - \tilde{P}_y^2 - \tilde{P}_y^2} \quad \text{Eq. (8)}$$

where ρ_0 is the radius of curvature. All components of the map, tangent map and tangent map derivative have been calculated. [15] We present the map here:

$$\bar{p}_1 = -\rho_0 \quad \text{Eq. (9a)}$$

$$+ (1 + \delta_s) \rho_0 \left\{ \sqrt{1 + \frac{2}{\beta_s} p_6 + p_6^2 - p_4^2 - \bar{p}_2^2} + p_2 \sin(\theta) - \left[\sqrt{1 + \frac{2}{\beta_s} p_6 + p_6^2 - p_4^2 - p_2^2} - \frac{1 + p_1/\rho_0}{1 + \delta_s} \right] \cos(\theta) \right\}$$

$$\bar{p}_2 = p_2 \cos(\theta) \quad \text{Eq. (9b)}$$

$$+ \left[\sqrt{1 + \frac{2}{\beta_s} p_6 + p_6^2 - p_4^2 - p_2^2} - \frac{1 + p_1/\rho_0}{1 + \delta_s} \right] \sin(\theta)$$

$$\bar{p}_3 = p_3 + (1 + \delta_s) \rho_0 p_4 \quad \text{Eq. (9c)}$$

$$\left\{ \theta + \sin^{-1} \left[p_2 / \sqrt{1 + \frac{2}{\beta_s} p_6 + p_6^2 - p_4^2} \right] \right.$$

$$\left. - \sin^{-1} \left[\bar{p}_2 / \sqrt{1 + \frac{2}{\beta_s} p_6 + p_6^2 - p_4^2} \right] \right\}$$

$$\bar{p}_4 = p_4 \quad \text{Eq. (9d)}$$

$$\bar{p}_5 = p_5 + \left(\frac{1 + \alpha \delta_s}{\beta_s} \right) \rho_0 \theta - (1 + \delta_s) \rho_0 \left(\frac{1}{\beta_s} + p_6 \right) \quad \text{Eq. (9e)}$$

$$\left\{ \theta + \sin^{-1} \left[\frac{p_2}{\sqrt{1 + \frac{2}{\beta_s} p_6 + p_6^2 - p_4^2}} \right] - \sin^{-1} \left[\frac{\bar{p}_2}{\sqrt{1 + \frac{2}{\beta_s} p_6 + p_6^2 - p_4^2}} \right] \right\}$$

$$\bar{p}_6 = p_6 \quad \text{Eq. (9f)}$$

where θ is the bend angle in radians.

A combined-function sector bend includes quadrupole and/or higher-order multipole fields along with the usual dipole field. The vector potential for these multipoles, which is required to define the Hamiltonian, must be obtained in the cylindrical coordinate system of the sector bend. This has been done in Ref.'s [16] and [17] for mid-plane symmetric fields (i.e. no skew multipoles). The results of Ref. [17] include the effects of a dipole field strength that varies along the bend arc. This vector potential is obtained as an expansion in x and y . In Ref. [17], these coefficients vary along the bend arc, if the dipole field is not constant.

The resulting Hamiltonian for a combined-function sector bend can be split into the sector bend Hamiltonian shown above and another exactly solvable piece, making

it well-suited to the Yoshida method. All components of the map (through fifth-order, which includes quadrupole, sextupole, octupole, decapole and duodecapole terms), tangent map and tangent map derivative for the combined-function piece of this Hamiltonian have been calculated, assuming a constant dipole field. [18]

An idealized sector bend does not include fringe fields, so one must implement an appropriate fringe field model before and after the element. Reference [16] presents both the third-order Lie polynomial and the corresponding second-order TRANSPORT coefficients for the dipole entrance and exit fringe fields. In fact, the map corresponding to the Lie polynomials of Ref. [16] can be calculated analytically, as can the components of the tangent map and tangent map derivative. [19] For example, the entrance fringe map is given by:

$$\bar{p}_1 = p_1 - f_{112} p_1^2 / 6 - f_{233} p_3^2 / 6 \quad \text{Eq. (10a)}$$

$$\bar{p}_2 = p_{yf} + R_{21} \bar{p}_1 \quad \text{Eq. (10b)}$$

$$\bar{p}_3 = p_3 - f_{134} p_1 p_3 / 6 \quad \text{Eq. (10c)}$$

$$\bar{p}_4 = p_{yf} + R_{43} \bar{p}_3 \quad \text{Eq. (10d)}$$

where p_{yf} and p_{yf} are defined in Eq.'s (11) below, and the f_{ijk} and R_{ij} terms refer to the entrance fringe Lie polynomial and R-matrix of Ref. [16].

$$p_{yf} = \frac{p_2 - \frac{1}{6} f_{134} p_1 p_2 + \frac{1}{6} f_{134} p_3 p_4 + \frac{1}{2} f_{111} p_1^2 + \frac{1}{6} f_{133} p_3^2 - \frac{1}{12} f_{111} f_{134} p_1^3 + \frac{1}{36} f_{133} f_{134} p_1 p_3^2}{1 - \frac{1}{6} (2f_{122} + f_{134}) p_1 + \frac{1}{18} f_{112} f_{134} p_1^2 - \frac{1}{18} f_{233} f_{134} p_3^2} ; \quad \text{Eq. (11a)}$$

$$p_{yf} = \frac{p_4 - \frac{1}{3} f_{133} p_1 p_3 - \frac{1}{3} f_{112} p_1 p_4 + \frac{1}{3} f_{233} p_2 p_3 + \left(\frac{1}{6} f_{111} f_{233} - \frac{1}{9} f_{112} f_{133} \right) p_1^2 p_3 + \frac{1}{18} f_{133} f_{233} p_3^3}{1 - \frac{1}{6} (2f_{122} + f_{134}) p_1 + \frac{1}{18} f_{112} f_{134} p_1^2 - \frac{1}{18} f_{233} f_{134} p_3^2} . \quad \text{Eq. (11b)}$$

5 ACKNOWLEDGMENTS

The authors express their gratitude to Alex Dragt, Etienne Forest and Michael Reusch for helpful discussions.

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