GENERATION OF EMITTANCE CONSERVING NON-KV DISTRIBUTIONS IN PERIODIC FOCUSING CHANNELS

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Abstract

In a uniform focusing beam transport system, an infinite variety of self-consistent phase space density distributions can be constructed. If these distributions are rms-matched to a periodic focusing channel, their self-consistent behavior is preserved if

1. both channels are equivalent in terms of their tunes,
2. the periodic lattice parameters are chosen to avoid the occurrence of structure resonances.

This result is verified by simulations assuming high current beam transport cases with $\sigma_0 = 60°$ zero current tune and $\sigma = 15°$ depressed tune. In the case of an interrupted solenoidal transport channel, all changes of the rms emittance are purely oscillating (non-growing). For the modulation of the matched beam envelope in this particular simulation, we obtain relative emittance fluctuation amplitudes of less than 1.00025.

1 INTRODUCTION

The question on whether strictly emittance conserving self-consistent beam transport through periodic focusing lattices is possible is still open[1]. Of course, this is an issue for long distance beam transport. Furthermore, analytical and numerical studies that calculate the time evolution of a deviation δf from an equilibrium distribution f depend on the existence of these equilibrium states.

Up to now, only the unphysical K-V distribution[2] is known to strictly conserve the beam's rms emittance with self-consistent space charge in non-continuous focusing channels. For all other phase space distributions representing the beam, the evolution of the beam envelope within the periodic focusing lattice is inevitably accompanied by a related variation of the rms emittance if the space charge effects are not negligible. The study of emittance conserving beam transport thus means to identify conditions under which these emittance variations are *periodically* oscillating.

2 CONTINUOUS CHANNELS

In this article, we restrict ourselves to unbunched, mono-energetic beams that propagate through linear focusing channels. As usual, the "trace space" notation is used, with the beam path length s instead of the time as the independent variable. If we furthermore exclude effects that originate in the actual charge granularity, i.e. if we use a continuous description of the beam's self-fields, Liouville's theorem applies for the 4-dimensional transverse phase space distribution f: $df/ds = 0$. A fictitious beam transport channel in which the external focusing forces act on the beam particles uniformly along the lattice can be classified as a scleronomic mechanical system. Under these circumstances, the equilibrium condition for f is simply given by a vanishing explicit time dependency:

$$f \text{ is stationary} \Longleftrightarrow \frac{\partial f}{\partial s} = 0 \; . \qquad (1)$$

With no explicit time dependency within the single particle Hamiltonian H, the particle's constant total energy is represented by H:

$$H(x,x',y,y') = \tfrac{1}{2} mc^2\beta^2\gamma \left(x'^2 + y'^2\right) + qV_{\text{eff}}(x,y) \; . \qquad (2)$$

Herein V_{eff} stands for the effective potential given by the sum of the assumed quadratic external focusing potential and the space charge potential:

$$qV_{\text{eff}}(x,y) = \tfrac{1}{2} mc^2\beta^2\gamma \left(k_x^2 x^2 + k_y^2 y^2\right) + \frac{q}{\gamma^2} V_{\text{sc}}(x,y) \; .$$

Because of Liouville's theorem, the equilibrium condition (1) is equivalent to a vanishing Poisson bracket ($[H,f] = 0$). This condition is fulfilled if f is a function of the energy H:

$$f = f(H) \; . \qquad (3)$$

In other words, f is stationary if the Isohamiltonians are surfaces of constant particle probability density. This state can be achieved for any phase space density function f, which means that an infinite variety of stationary phase space density functions exists.

3 PERIODIC FOCUSING CHANNELS

Non-continuous focusing systems are characterized by the fact that the focusing elements maintaining the transverse beam extend are spatially separated by intermediate drift spaces. Under these conditions, no such thing as a "stationary phase space density" exists and Eq. (1) no longer applies. The question under which conditions the beam quality is sustained thus appears in a more general form. For periodic focusing channels we must ask whether strictly *periodic* solutions for the beam states exist. A possible way to answer this question is based on the idea to relate the particle motion within periodic and continuous focusing systems by a canonical transformation. This transformation must map the Hamiltonian of the continuous focusing channel onto the appropriate Hamiltonian for the periodic channel. If we then express the Hamiltonian for the continuous focusing channel – which represents the particle energies as constants of motion – in terms of the new variables, we obtain the constants of motion that exist within the periodic focusing system.

4 RELATING CONTINUOUS AND PERIODIC FOCUSING CHANNELS

4.1 K-V Beams

As a special case of Eq. (3), the K-V phase space density function[2] is defined by the property that all beam particles possess the same single particle energy H_0. Mathematically, this property can be expressed in terms of a "δ-function":

$$f(H) \propto \delta(H - H_0) \; .$$

It turns out that exactly this density function is associated with strictly linear self-fields. Together with the linear focusing forces, the entire single particle beam dynamics are thus governed by linear equations of motion. The associated Hamiltonian (2) then contains a space charge potential $V_{\mathrm{sc}}(x,y)$ that is a strictly quadratic function of the spatial coordinates x and y. Under these special conditions, the mapping transformation that relates the beam dynamics in continuous and non-continuous focusing channels is also linear. In order to obtain a more transparent formulation, this transformation is split into two steps. The intermediate particle coordinates after the first transformation step will be marked by a tilde. Similarly, all quantities that appear after the second step will be marked by a bar. The first linear transformation can be written in matrix notation as

$$\begin{pmatrix} x_i \\ x_i' \end{pmatrix} = \begin{pmatrix} \cos\Psi_x(s) & -\beta_x \sin\Psi_x(s) \\ \beta_x^{-1}\sin\Psi_x(s) & \cos\Psi_x(s) \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{x}_i' \end{pmatrix} \tag{4}$$

with $\beta_x \equiv \gamma_x^{-1} = \mathrm{const.}, \alpha_x \equiv 0$ the Courant-Snyder[3] functions given for a matched beam with self-consistent space charge forces within the constant focusing channel, and $\Psi_x(s)$ defined as the difference of the phase advances between both systems as a function of the position s along the beam lines

$$\Psi_x(s) = \bar{\sigma}_x(s) - \sigma_x(s) = \int_{s_0}^{s} \frac{dz}{\bar{\beta}_x(z)} - \frac{s - s_0}{\beta_x} \; .$$

The first transformation can be interpreted as a *shifting* of all beam particles by an axial distance $\ell_x(s) = \beta_x \Psi_x(s)$ within the continuous focusing channel. The second canonical transformation – the "matching transformation" – is then applied to the beam at equal phase advance rather than at equal longitudinal position s as:

$$\begin{pmatrix} \tilde{x}_i \\ \tilde{x}_i' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_x/\bar{\beta}_x(s)} & 0 \\ \bar{\alpha}_x(s)/\sqrt{\beta_x\bar{\beta}_x(s)} & \sqrt{\bar{\beta}_x(s)/\beta_x} \end{pmatrix} \begin{pmatrix} \bar{x}_i \\ \bar{x}_i' \end{pmatrix} \tag{5}$$

We note that the total transformation defined by Eqs. (4) and (5) is just a special case of a more general mapping theory developed for alternating-gradient systems[3]. The Hamiltonian (2) canonically transformed via (4) and (5) is obtained as

$$\bar{H}(\bar{x},\bar{x}',\bar{y},\bar{y}') = \tfrac{1}{2} mc^2\beta^2\gamma \left(\bar{x}'^2 + \bar{y}'^2\right) + q\bar{V}_{\mathrm{eff}}(\bar{x},\bar{y};s) \; . \tag{6}$$

This Hamiltonian (6) exactly describes a K-V beam within a non-continuous focusing channel – with the quadratic effective potential now being explicitly s-dependent:

$$q\bar{V}_{\mathrm{eff}} = \tfrac{1}{2} mc^2\beta^2\gamma \left(\bar{k}_x^2(s)\bar{x}^2 + \bar{k}_y^2(s)\bar{y}^2\right) + \frac{q}{\gamma^2} V_{\mathrm{sc}}(\bar{x},\bar{y};s) \; .$$

Here, $\bar{k}_x^2(s)$ and $\bar{k}_y^2(s)$ represent the external focusing forces that act on the beam as functions of s. From Eq. (2) we observe immediately that I_i is a constant of motion for each particle i propagating within a matched K-V beam in a continuous focusing channel:

$$I_i^{\mathrm{KV}} = \beta_x \, x_i'^2 + \gamma_x \, x_i^2 \; . \tag{7}$$

It is easily shown by inserting (4) and (5) into (7) that this quantity expressed in the particle coordinates pertaining to the non-continuous focusing system becomes:

$$I_i^{\mathrm{KV}} = \bar{\beta}_x(s)\, \bar{x}_i'^2 + 2\bar{\alpha}_x(s)\, \bar{x}_i \bar{x}_i' + \bar{\gamma}_x(s)\, \bar{x}_i^2 \; . \tag{8}$$

Eq. (8) is a conserved quantity for all particles within a K-V beam. It is readily identified as the "single particle emittance".

We remark that the strictly isomorphic behavior of the particle motion within K-V beams between continuous and non-continuous focusing channels follows from the equivalence of the time-dependent and the time-independent harmonic oscillators[4, 5]. It explains the success of the so-called "smooth approximation"[6]. As we observe, these types of channels are not only approximately but exactly equivalent within the K-V model.

4.2 non-K-V Beams

We now want to generalize the transformation theory outlined in the previous subsection in order to cover all stationary phase space density functions conforming to (3). For non-K-V beams, the space charge potential functions are no longer purely quadratic, which means that the particle equations of motion are no longer linear. Therefore, we cannot write the integral of these equations in a closed form as given by the solution matrix (4), which represents the shifting transformation in the linear case. We are thus forced to restrict ourselves in generalizing the shifting transformation (4) to an infinitesimal axial step. We hereby correlate the beams within both system only over an infinitesimal distance $\delta\ell_x(s)$ along the lattice. As a consequence, we cannot expect anymore to obtain a conserved quantity that applies for finite steps. Rather, the invariant appears as a sum of infinitesimal quantities, i.e. as a differential equation.

The canonical transformation that moves the particles an infinitesimal axial step is generated by the Hamiltonian (2) itself. The coordinates of the new system are obtained by:

$$x_i = \tilde{x}_i - \delta\ell_x(s)\, \tilde{x}_i' \quad , \quad x_i' = \tilde{x}_i' + \delta\ell_x(s)\, \kappa_x^2\, \tilde{x}_i \tag{9}$$

with $\delta\ell(s) = [\beta_x/\bar{\beta}_x(s) - 1]\,\delta s$ and κ_x^2 defined as

$$\kappa_x^2 = \beta_x^{-2} - \frac{q}{mc^2\beta^2\gamma^3} \frac{E_x(\tilde{x}_i,\tilde{y}_i) - E_x^{\mathrm{KV}}(\tilde{x}_i,\tilde{y}_i)}{\tilde{x}_i} \; .$$

In this equation, $E_x - E_x^{\mathrm{KV}}$ denotes the difference of the actual space charge field to the linear field function of an equivalent K-V beam. It is easily verified that (9) is the infinitesimal limit of (4) if the underlying phase space density function is of the K-V type. We insert (9) and (5) subsequently into the time-independent Hamiltonian (2). After summing over all N particles of the beam, this "invariant" reads

$$\frac{d\bar{\varepsilon}_x^2(s)}{\bar{\varepsilon}_x(s)\bar{\beta}_x(s)} + \frac{d\bar{\varepsilon}_y^2(s)}{\bar{\varepsilon}_y(s)\bar{\beta}_y(s)} + \frac{2}{mc^2\beta^2\gamma^3 N} d(\bar{W} - \bar{W}^{\mathrm{KV}}) = 0 \qquad (10)$$

Here, $\bar{\varepsilon}_x(s)$ and $\bar{\varepsilon}_y(s)$ denote the s-dependent rms emittances in the x, x'- and y, y'-phase space planes within the non-continuous focusing system. The space charge potential terms of all particles sum up to yield the field energy $\bar{W}$ of the actual beam. $\bar{W} - \bar{W}^{\mathrm{KV}}$ thus provides us with the *excess* field energy the actual beam possesses in *addition* to the equivalent K-V beam. Eq. (10) has been obtained earlier by Wangler et. al.[7] in an alternative derivation.

5 SIMULATION RESULTS

The results of the previous sections can be used to explain the results of computer simulations of charged particle beams. If we advance a K-V beam through a periodic solenoidal focusing lattice, we observe in Fig. 1 that its rms emittance is indeed a conserved quantity – in agreement with Eq. (8). For non-K-V phase space distributions, Eq. (10) states that the emittance varies with the modulation of the beam envelopes. This statement is confirmed by the dashed line in Fig. 1, which shows the evolution of the rms emittance of an initial "water bag" distribution. To lowest order, the emittance oscillates with the period of the focusing lattice. Nevertheless, higher order oscillation modes are also present. As the long term simulations displayed in Fig. 2 show, the emittance fluctuations do not lead to an overall emittance growth – at least for the periodic solenoid case. In the periodic quadrupole channel simulation, we observe a non-saturating growth of the rms emittance. This behavior can be attributed to non-Liouvillean temperature balancing effects[8] caused by the charge granularity. This effect occurs even more pronounced in computer simulations because of the enhanced granularity associated with the macro-particle concept.

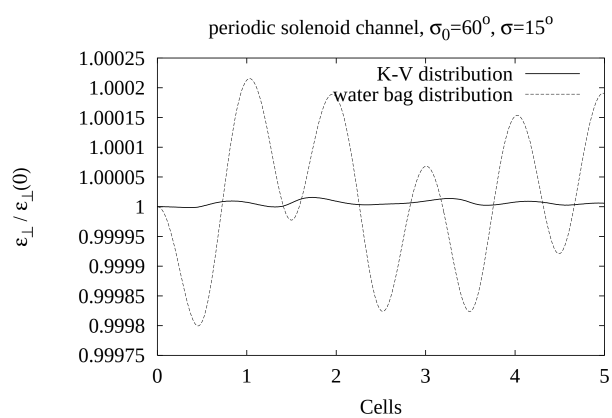

Figure 1: Transverse emittance functions within a periodic solenoid channel for K-V and "water bag" phase space density profiles with $\sigma_0 = 60^\circ$ and $\sigma = 15^\circ$.

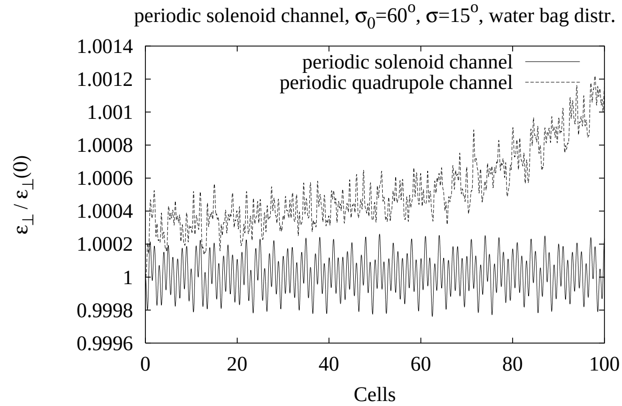

Figure 2: Long term transverse emittance evolution within periodic focusing channels for an initial "water bag" phase space density profile with $\sigma_0 = 60^\circ$ and $\sigma = 15^\circ$.

6 CONCLUSIONS

A method to minimize emittance growth effects within beam transport periodic channels can be sketched by a simple "cookbook recipe". For a given periodic focusing channel producing a zero current tune σ_0, and a given beam characterized by its emittance and current, we generate a self-consistent distribution with a specific form of $f(H)$ with the same emittance and current for the equivalent continuous focusing channel, i.e. a channel that produces the same zero current phase advance σ_0 over the length that corresponds to the focusing period of the periodic channel. We then match this phase space distribution according to (5) to the given periodic lattice. If we avoid a regime where structure resonances exist, we thus obtain a "quasi stationary" beam behavior in the periodic channel.

7 REFERENCES

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