# BETATRON MOTION WITH LOGARITHM–LIKE PERTURBATION IN STORAGE RINGS

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*Abstract*

Logarithm–like perturbation of the transverse potential occurs if the amplitude of betatron oscillations of a beam particle exceeds the beam (or the ion core for electron storage rings) radius. This situation takes place in storage rings with intense circulating beams for multiturn injecting particles, for the particles producing a halo, and for a vast part of beam electrons if electrostatic ion clearing electrodes applied. The closed analytical expression for the response of betatron tunes on the amplitude of oscillation is presented. Also transverse couple resonances driven by the perturbation are estimated. It is shown that the perturbation causes nonlinear coupling of the transverse degrees of freedom. Recommendations on choose a proper working point are made.

## 1 INTRODUCTION

Some modes of operation of storage rings with intense beams involve perturbation of the transverse focusing by logarithm-like potential. These cases are:

- multiple injection, especially at low energy as in the compact synchrotron light factories and Compton sources;
- operation of a storage ring with ion clearing by electrostatic electrodes.

In these cases amplitudes of transverse oscillations of some particles (the injecting ones in the first case and peripherical — in the second) sufficiantly exceed transverse dimensions of the source of perturbative electric field, e.g., the ion core [1] or the stored beam.

## 2 MODEL

The following model is considered. Parabolic focusing potential in which beam particles are oscillating, are perturbed by the potential of a uniformly charged rod placed in the axis of a round conductive chamber. This rod represents the space charge of the beam or the ion core.

Suppose density distribution possesses the form:

$$\rho(r) = n(r)e = en_0\,[\mathrm{H}(r) - \mathrm{H}(r-a)] \qquad (1)$$

where $\mathrm{H}(r)$ is the Heaviside step function, $e$ — electron charge, $n_0$ — density of the beam or the ion core, $a$ — beam (core) radius.

Potential function $\Phi$ of this system with the natural border conditions $\Phi'(r=0) = 0$ and $\Phi(b) = 0$ ($b$ is the pipe radius) has a form:

$$\Phi(r) = \frac{Ne}{4\pi\epsilon_0}\left\{1 + \ln\frac{b^2}{a^2} - \mathrm{H}(r-a)\left(1 + \ln\frac{r^2}{a^2}\right) - \frac{r^2}{a^2}\left[\mathrm{H}(r) - \mathrm{H}(r-a)\right]\right\}. \qquad (2)$$

Here $N = \pi a^2 n$ is the longitudinal charged density of the rod; $\epsilon_0$ — permittivity of vacuum.

The potential function (2) has its extreme value at the beam axis $r = 0$:

$$\max|\Phi(r)| = |\Phi(0)| = \frac{N\,|e|}{4\pi\epsilon_0}\left\{1 + \ln\frac{b^2}{a^2}\right\}, \qquad (3)$$

whereas field strength reaches its maximal value at the beam edge:

$$\max|\Phi'(r)| = |\Phi'(a)| = \frac{N\,|e|}{2\pi\epsilon_0 a} \qquad (4)$$

The potential function $\Phi(r)$ is depicted in Fig.1.

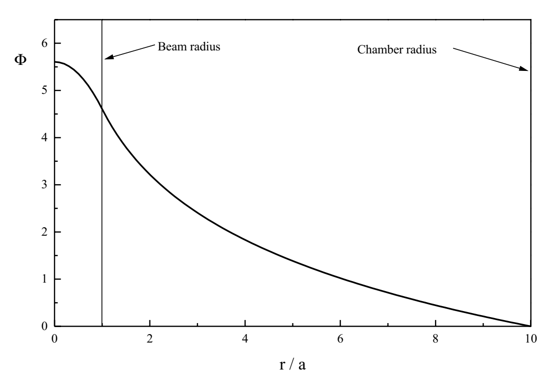


Figure 1: Repulsive potential of the uniformly charged beam in a round chamber.

As is seen from this plot and from expression (2), The potential function has parabolic form within the rod and the logarithmic one — beyond it. This potential would cause the linear shift of the betatron number $Q$ for oscillations which amplitudes do not exceed the rod radius and the nonlinear response on frequency for the larger amplitudes.

## 3 PERTURBATION OF THE BETATRON NUMBERS

Common procedure of getting amplitude response of the betatron frequency is as follows. Let start from the unperturbed Hamilton function of betatron motion of the form [2]:

$$\mathrm{U}_0(I_x, I_z, \varphi_x, \varphi_z; \vartheta) = I_x Q_{x0} + I_z Q_{z0} \tag{5}$$

with $I_y$ related to averaged over the ring circumference transverse coordinates as

$$y = \sqrt{\frac{2I_y}{Q_y}} \cos\left(Q_y \vartheta\right), \tag{6}$$

where $y = (x, z)$ is a transverse coordinate; $\vartheta$, the azimuthal angle coordinate.

Perturbation of Hamiltonian (5) by electric field is

$$\mathrm{U} = \mathrm{U}_0 + \mathrm{U}_1 = \mathrm{U}_0 + \frac{R^2 e}{m_0 c^2 \sqrt{\gamma^2 - 1}} \Phi \tag{7}$$

Taking into account $r^2 = x^2 + z^2$ and substituting (6) into (2) and afterwards to (7), we get the perturbed Hamiltonian as $\mathrm{U} = \mathrm{U}(I, Q, \vartheta)$.

To the first approximation in perturbation from (7) we get:

$$Q_x(I_x, I_y = 0) = \left\langle \frac{\partial \mathrm{U}}{\partial I_x} \right\rangle \tag{8}$$

where angle brackets show averaging over the 'fast' angle $Q_y \vartheta$.

Thus, electrostatic potential (2) causes amplitude dependence of the betatron number as:

$$\begin{aligned} Q_x(I_x, I_y &= 0) = Q_{x0} + \\ &\frac{e}{|e|} \frac{N r_0 R^2}{Q_{x0} a^2 \sqrt{\gamma^2 - 1}} \mathrm{F}\left(\frac{2I_x}{Q_{x0} a^2}\right), \end{aligned} \tag{9}$$

$$\mathrm{F}(y) \equiv 1 + \frac{2\mathrm{H}(y-1)}{\pi} \left(\frac{1}{y} - 1\right) \arccos \frac{1}{\sqrt{y}}.$$

Here $r_0$ is the classical radius of a beam particle; $\gamma$, the relativistic factor; $R$, the average machine radius.

The function $\mathrm{F}(y)$ containing the $Q$–dependence upon the amplitude is plotted in Fig.2.

The defined function $\mathrm{F}(y)$ amplitude has the continuous first derivative at the rod radius, $y = 1$:

$$\frac{\mathrm{dF}}{\mathrm{d}y} = -\frac{2}{\pi}(y-1)^{3/2} + \mathrm{O}(y-1)^{5/2}.$$

As it could be seen from the Fig. 2, the $Q$–shift is constant within the rod, then the betatron frequency decreases with increasing of the amplitude.

## 4 TRANSVERSE RESONANCES

As it is well known (see, e.g., [2]), developing of a resonance requires meeting of the following conditions:

- the working point being close to the resonant line, $mQ_x + nQ_z + p = \delta \ll 1$ ($m, n, p$, integers)
- the corresponding perturbation force with proper azimuthal Fourier component presenting in the ring orbit.

In other words, the resonance must have the stop band, and the working point $Q_x, Q_z)$ must be within it.

The space charge force of the rod moves og the working point $Q_x, Q_z$, so it can reach the stop band of a natural resonance. Moreover, the space charge itself can cause the resonant perturbation [3]. The cpace charge resonances are similar to the beam crossing resonances being studied intensively. The resonances due to the space charge forces are not so complicated for investigation because of the coasting nature of the beam. As it has been shown in [3], the ion core drives the nonlinear difference resonances

$$2(mQ_x - nQ_z) = \delta \ll 1.$$

These resonances capture the peripheral beam particles. The 'transverse energy' of these particles is the constant of motion:

$$E_\perp = I_x Q_x^2 + I_z Q_z^2 = \text{const} \tag{10}$$

It leads to occurence of the halo around the beam and can cause the 'resonant' increase in the beam losses when the halo tails reach the aperture of the ring. These resonances may be harmful for a machine with the low-energy multiple injection, where the injected beam with a large amplitude experiences the nonlinear forces due to the ions confined by the circulating beam or due to the dense beam itself. Especially it concerns the rings with $Q_z < Q_x$. Increase in a value of the relation $Q_x/Q_z$ will lead to enlarging of the halo.

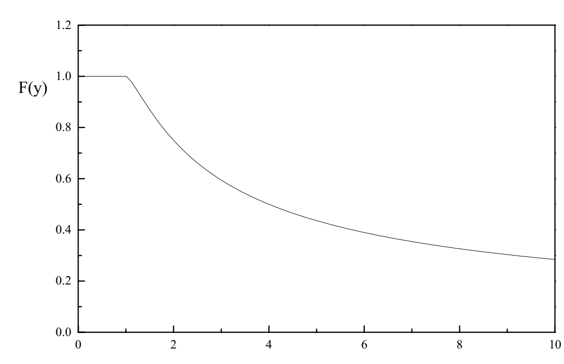


Figure 2: $Q$–Dependence on the squared relative amplitude.

## 5 REFERENCES


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[3] E. Bulyak, V. Gonchar, and V. Kurilko,“Dependence of the beam emittance on the beam current in an electron storage ring,” Proc. 13 ICHEA, Novosibirsk, 1986, Vol. 2, p. 174, 1987.