Oak Ridge National Laboratory

Beams with Three-Fold Rotational Symmetry: A Theoretical Study

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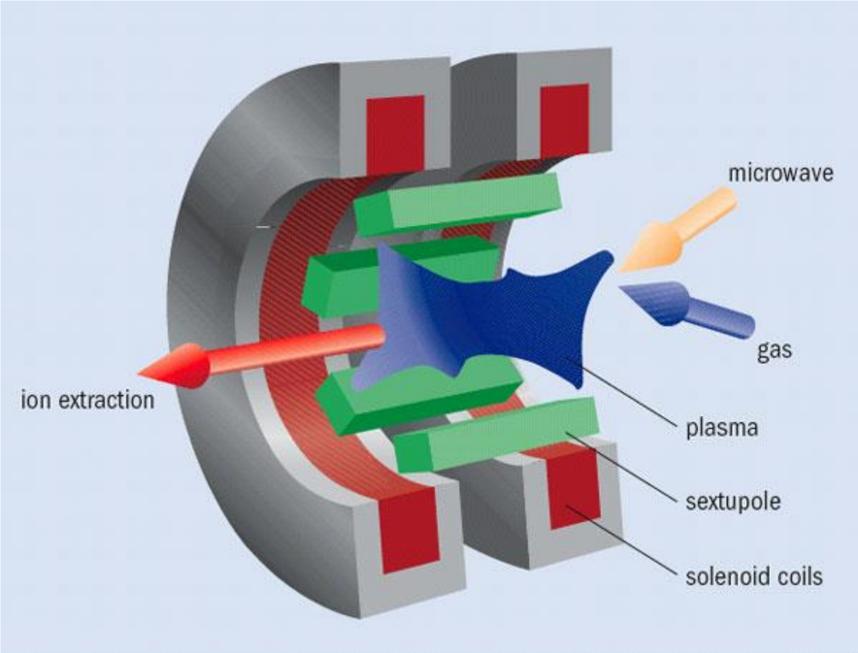
*Previously from NSCL/MSU which provided significant support to this work

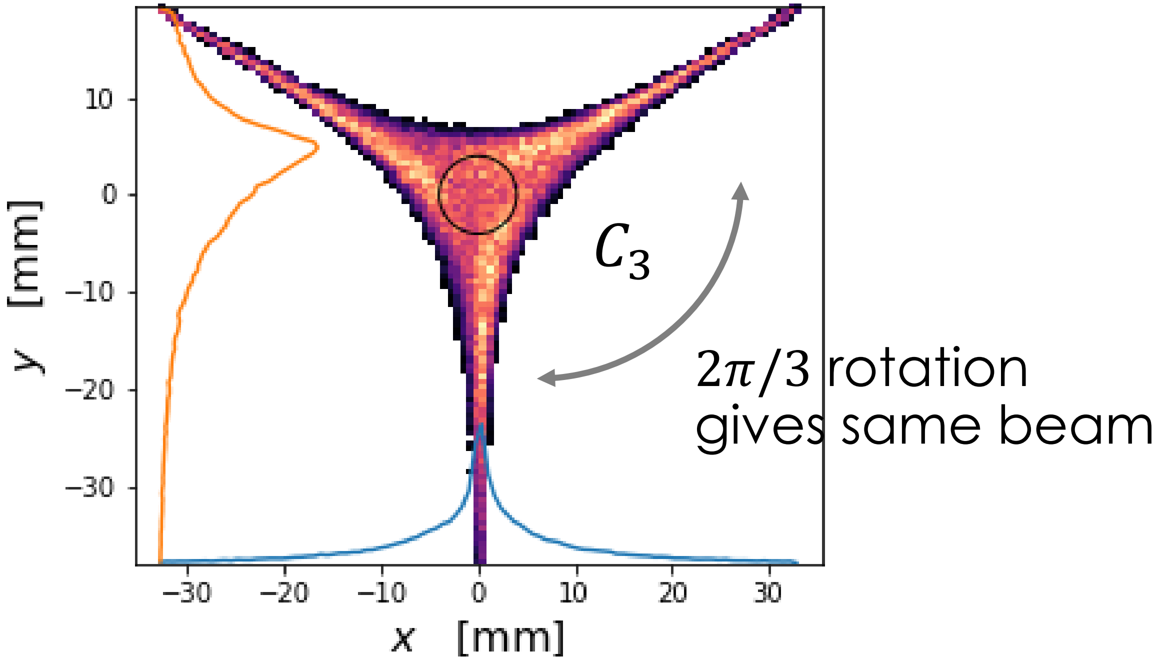
Beam Inherits Rotational Symmetry of Source / Beam Line

- True in idealized system
  - System identical in different rotated transverse coordinate systems
  - Violated by misalignments and nonideal elements
- Two symmetries that often occur:
  - Cylindrical symmetry
    - Solenoids, einzel lens
  - Two-fold rotational symmetry
    - Quadrupole transport
- Notation
  - C_n: n-fold discrete rotational symmetry
  - SO(2): continuous rotational symmetry (also called axisymmetry)

ECR Sextupole Imposes C_3 Symmetry on Beams



[Daniela Leitner (LBNL), CERN Courier]



[ECR simulation results at extraction plane, courtesy of Vladimir Mironov (JINR)]

- Beam envelope and emittance depend on choice of axis?
- x-, y-emittances unequal and change upon coupling in solenoid transport?

Rotational Symmetry Imposes Constraints on Transverse Beam Moments

- Rotational symmetry: there are angles θ for which beam moments are invariant under the transformation

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \equiv \mathbf{R} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$$

- For C_n beam, $\theta = 2k\pi/n$
- For SO(2) beam, any θ

- Same transformation in complex coordinates

$$\begin{pmatrix} w \\ \overline{w} \\ w' \\ \overline{w'} \end{pmatrix} \equiv \begin{pmatrix} x + iy \\ x - iy \\ x' + iy' \\ x' - iy' \end{pmatrix} \qquad \begin{pmatrix} w \\ \overline{w} \\ w' \\ \overline{w'} \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} w \\ \overline{w} \\ w' \\ \overline{w'} \end{pmatrix}$$

Deriving Constraints via Complex Moments

- Rotational symmetry: $\left\langle w^{a_1}\overline{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\right\rangle = e^{i(a_1-a_2+a_3-a_4)\theta}\left\langle w^{a_1}\overline{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\right\rangle$

$$e^{i(a_1-a_2+a_3-a_4)\theta} \neq 1 \longrightarrow \left\langle w^{a_1}\overline{w}^{a_2}w'^{a_3}\overline{w'}^{a_4}\right\rangle = 0$$ (two constraints)

- For $\theta = 2\pi/3$

$$\mathrm{Re}\left(\langle ww\rangle\right) = 0 \quad \Rightarrow \quad \langle xx\rangle = \langle yy\rangle \qquad \mathrm{Im}\left(\langle ww\rangle\right) = 0 \quad \Rightarrow \quad \langle xy\rangle = 0$$

$$\mathrm{Re}\left(\langle ww'\rangle\right) = 0 \quad \Rightarrow \quad \langle xx'\rangle = \langle yy'\rangle \qquad \mathrm{Im}\left(\langle ww'\rangle\right) = 0 \quad \Rightarrow \quad \langle xy'\rangle = -\langle x'y\rangle$$

$$\mathrm{Re}\left(\langle w'w'\rangle\right) = 0 \quad \Rightarrow \quad \langle x'x'\rangle = \langle y'y'\rangle \qquad \mathrm{Im}\left(\langle w'w'\rangle\right) = 0 \quad \Rightarrow \quad \langle x'y'\rangle = 0$$

- Detailed treatment: **Moment constraints in beams with discrete and continuous rotational symmetry**

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(PRAB, to be submitted)

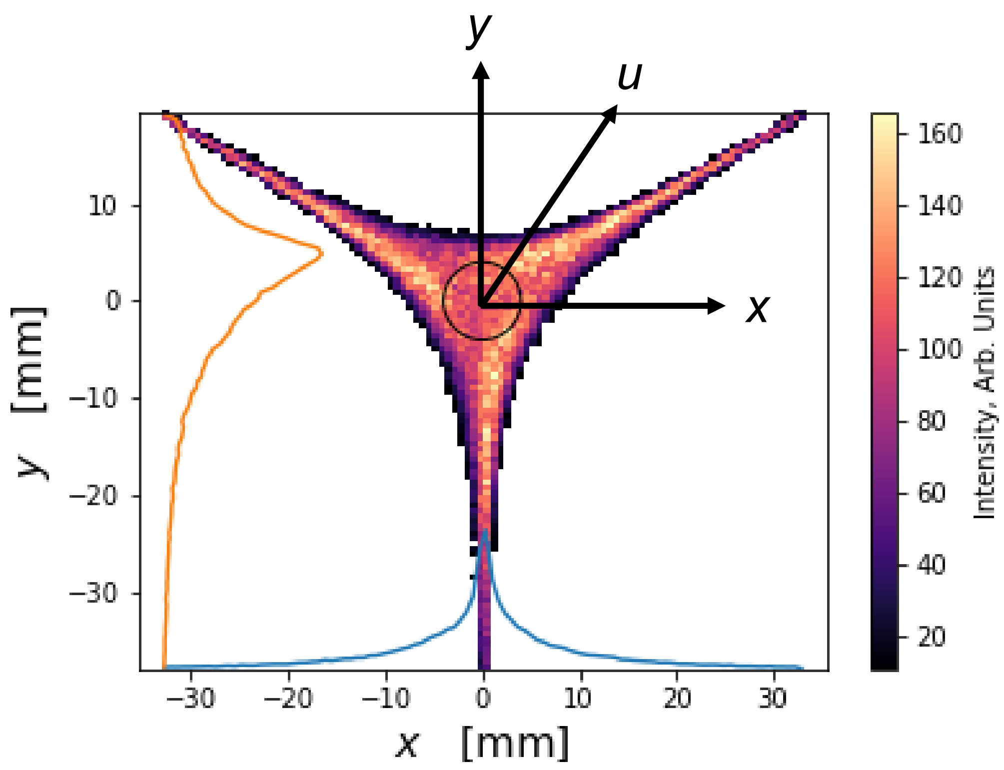
C_3 Beams are Effectively Axisymmetric

Same set of 2nd order moment constraints as axisymmetric beam

$$\langle xx \rangle = \langle yy \rangle \qquad \langle xy \rangle = 0$$

$$\langle xx' \rangle = \langle yy' \rangle \qquad \langle xy' \rangle = -\langle x'y \rangle$$

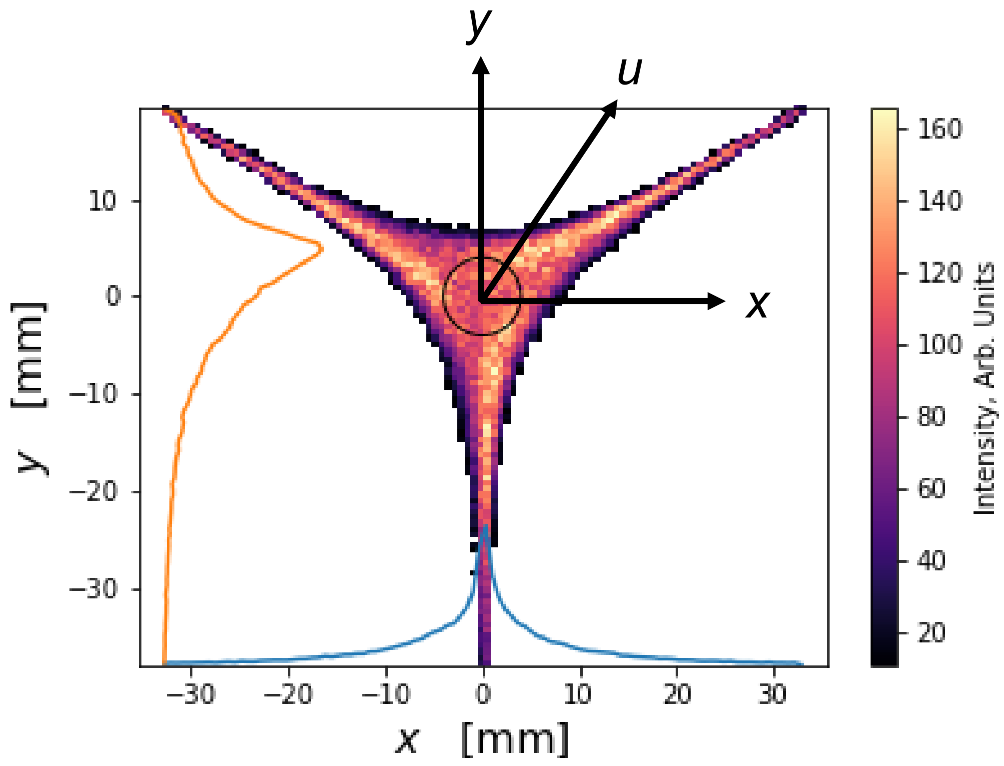
$$\langle x'x' \rangle = \langle y'y' \rangle \qquad \langle x'y' \rangle = 0$$



- C_3 beams and SO(2) beams are indistinguishable in terms of 2nd order moments
  - Their 2nd order moments have the exact same properties
  - Distinguishable in terms of higher order moments

C_3 Beams have Identical RMS Envelope and Emittance along Any Direction

$$u(\theta) = x\cos\theta + y\sin\theta$$



- Properties of 2nd order moments of SO(2) beam (which are shared by C_3 beams):
  - $\langle uu \rangle = \langle xx \rangle$
  - $\langle uu' \rangle = \langle xx' \rangle$
  - $\langle u'u' \rangle = \langle x'x' \rangle$

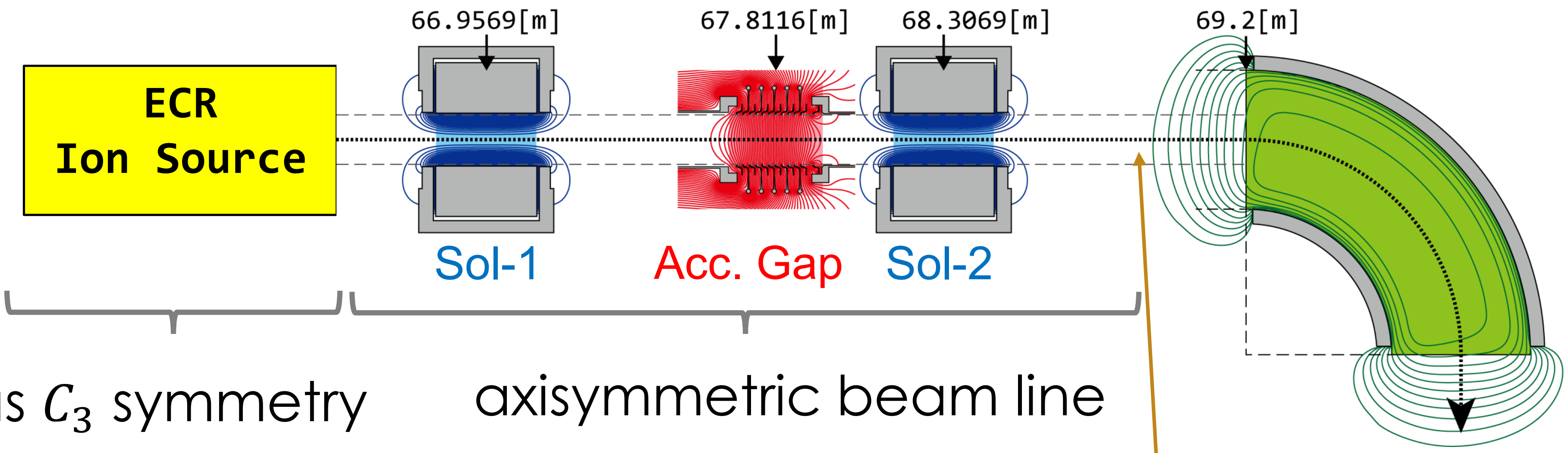
  For all θ

- Projected phase space distribution in u-u' phase space changes significant with θ, but rms phase space ellipse remain identical

- Consequence of symmetry alone
  - True with multi-species space charge, chromatic aberrations, radial field nonlinearities

Application to FRIB Front End Artemis Beamline

Front section of FRIB beam line (Image courtesy of Kei Fukushima)



ECR has C_3 symmetry

axisymmetric beam line

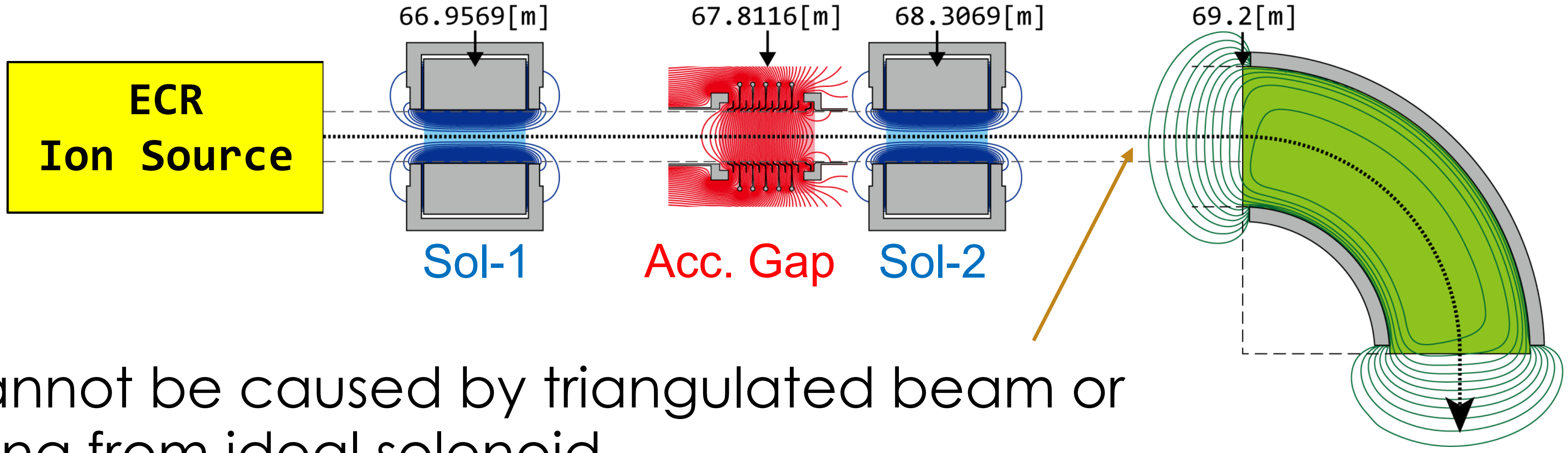
x, y phase space scanners

- Beam should have C_3 symmetry up to non-zero dipole field

- Back projection found $\varepsilon_x \neq \varepsilon_y$ before dipole
  - Sign and magnitude depends on solenoid strengths

Cause of $\varepsilon_x \neq \varepsilon_y$ is Broken Symmetry

Front section of FRIB beam line (Image courtesy of Kei Fukushima)



x, y phase space scanners

- $\varepsilon_x \neq \varepsilon_y$ cannot be caused by triangulated beam or x-y coupling from ideal solenoid
- Only one fundamental cause: broken symmetry
- Motivated search for source of broken symmetry
  - Suggested by ECR Team: Sol-1 has strong multipole fields due to leads design
  - Proved by Alexander Plastun via magnet simulations

Constraints on 3rd Order Moments of C_3 Beams

- Use same technique to derive 12 constraints

$$\langle xxx \rangle = -\langle xyy \rangle$$
$$\langle yyy \rangle = -\langle xxy \rangle$$
$$\langle x'x'x' \rangle = -\langle x'y'y' \rangle$$
$$\langle y'y'y' \rangle = -\langle x'x'y' \rangle$$

$$\langle xyy' \rangle = \langle x'yy \rangle = -\langle xxx' \rangle$$
$$\langle xx'y \rangle = \langle xxy' \rangle = -\langle yyy' \rangle$$
$$\langle x'yy' \rangle = \langle xy'y' \rangle = -\langle xx'x' \rangle$$
$$\langle xx'y' \rangle = \langle x'x'y \rangle = -\langle yy'y' \rangle$$

- In comparison, all 3rd order moments vanish for C_2 and SO(2) beams

Conclusion

- C_3 symmetry guarantees equal x, y envelopes and emittances
  - 2nd order moments have same properties as those of SO(2) beam
  - Failure to hold suggests broken symmetry (e.g. misalignments, nonideal elements)

- Theoretical results successfully clarified beam dynamics at FRIB
  - Arguments derived purely from symmetry

- Benchmark ECR simulations?
  - Consistent with model provided by Vladimir Mironov (JINR) in preliminary comparisons