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SPACE-CHARGE POTENTIAL FOR ELLIPTICAL BEAMS*

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Abstract

This work is motivated by the weak-strong beam-beam effect as occurs in colliding charged-particle beams. We consider beams with elliptical cross section and power law binomial forms for the density distribution. We demonstrate explicitly how to construct analytically the space-charge potential inside the "strong" beam. This is essential to the program of calculating beam-beam effects for non-gaussian beams.

INTRODUCTION

The calculation of the electrostatic force on a test particle within a charge distribution of circular or elliptical cross section has a long history, starting with James Clerk Maxwell [1], as does the uniform ellipsoid (3D). These methods were reported by Klein [2] and Kellog [3], and later generalized by Houssais [4]. Some of the simpler cases have been used (Kapchinski & Vladimirski [5], and Sacherer [6]) in accelerator physics since the 1960s. Keil [7], Montague [8], Zotter [9], Bassetti & Erskine [10] all started with the 2D Housais potential for a Gaussian. Nevertheless, except for the uniform and Gaussian [4] elliptic beam, the method is not widely known.

Here we show explicitly how the technique is employed for power law binomial beams, and how our results relate to the gaussian beam through the second moments.

Relation to Beam-Beam Effects

Assuming that the beam has a Gaussian charge distribution, Houssais and later Bassetti & Erskine, took the transverse density to be Exp[$-R^2/2$] where R^2 = [$(x/sx)^2 + (y/sy)^2$]. Here *x,y* are Cartesian coordinates and sx,sy are the standard deviation of the distribution.

Particle beams are not necessarily Gaussian. We take density in the quadratic form $[1-(R/b)^2]^N$. With radius *b* and power index *N* suitably chosen, this can approximate the Gaussian increasingly well, even for relatively small *N*. Moreover, when we calculate the corresponding potential, for R>4 σ or larger, it differs very little from the Gaussian case; because the residual charge beyond radius R is of order Exp[$-R^2/2$] <<< 1 leaving only the Log[R] term, which is common to both potentials. Assuming the 2-dimensional Green's function, we show how to construct the potential leading to explicit and finite series in x and y. Finally, we illustrate for the case *N*=6, b=4σ and write the low order terms in x and y for sigma-x not equal sigma-y.

Normalization & Equal Variance

The 2D and 3D Gaussian distributions are the product of 1D distributions of the form:

$$Exp[-(1/2)(x/s)^\wedge 2\,]/(\sqrt{2\pi}\, s)$$

where *s* is the standard deviation or r.m.s value of x (i.e. s^2 is variance).

Consider now the hard-edge elliptical distributions

$$\rho = [1 - Z(0)]^N / I_N$$

where integer N is the power law, and

$$Z[t] = \frac{x^2}{\mathrm{sx}^2 + t} + \frac{y^2}{\mathrm{sy}^2 + t}$$

The normalization integral is

$$I_N = \pi \mathrm{sx}\,\mathrm{sy}/(1+N)$$

The hard edge distribution will have the same variance as the Gaussian, if the semi-axes of the ellipse sx, sy are chosen according to the form

$$< x^2 > = s^2 = \frac{\mathrm{sx}^2}{4 + 2N}$$

POTENTIAL FUNCTION

Inside the particle beam, the potential, *V*, is the solution of Poisson's equation $-\nabla^2 V = \rho$. From the treatises of Kellog and Houssais, it follows that *V* is of the form:

$$V[Z] = \frac{I_N}{4\pi} \int_0^\infty \frac{(1 - Z[t])\rho[Z[t]]}{S[t]} dt$$

where $S[t] = \sqrt{(\mathrm{sx}^2 + t)(\mathrm{sy}^2 + t)}$. Here *t* is a dummy variable; it is not a time coordinate. The integral is logarithmically divergent at Z=0, that is when x=y=0. The potential is defined up to an arbitrary constant. Hence we subtract V[Z=0] to eliminate the singularity. Thus:

$$V[Z] = \frac{I_N}{4\pi} \int_0^\infty \frac{-1 + (1 - Z[t])\rho[Z[t]]}{S[t]} dt$$

$$V[Z] = \frac{1}{4\pi} \int_0^\infty \frac{-1 + (1 - Z[t])^{1+N}}{S[t]} dt$$

Three-Dimensional Case

The 3D case is a simple extension of the formulas above:

$$Z[t] = \frac{x^2}{\mathrm{sx}^2 + t} + \frac{y^2}{\mathrm{sy}^2 + t} + \frac{z^2}{\mathrm{sz}^2 + t}$$

$$S[t] = \sqrt{(\mathrm{sx}^2 + t)(\mathrm{sy}^2 + t)(\mathrm{sz}^2 + t)}$$

with revisions of the normalization constant I_N and variance formula. Technically, it is not necessary to subtract off V[Z=0] because the integral is not divergent at x=y=z=0. However, it is "handy" to have the potential function equal zero at the origin.

Except in the cases N=0 (constant density) and N=∞ (Gaussian), the 3D integral cannot be obtained analytically, and numerical integration of elliptic integrals must be resorted to; so, we shall say no more about the 3D ellipsoidal charge distribution.

2D CONSTANT DENSITY, N=0

The uniformly filled ellipse with boundary (1-Z[0])=0. The integral over *t* is performed giving the expression:

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$$V[x,y] = -\frac{\mathrm{sy}x^2 + \mathrm{sx}y^2}{2\pi\ \mathrm{sx\ sy}\ (\mathrm{sx}+\mathrm{sy})}$$

Forming the gradient, **E** = - Grad V, gives the field components:

$$\{\mathrm{Ex}, \mathrm{Ey}\} = \frac{1}{\pi(\mathrm{sx}+\mathrm{sy})}\left\{\frac{x}{\mathrm{sx}}, \frac{y}{\mathrm{sy}}\right\}$$

The field rises linearly from the origin, as is well known. Forming the Laplacian, we recover the normalized charge density $\rho[x,y] = 1/\pi\mathrm{sxsy}$.

2D QUADRATIC DENSITY, N=1

Setting index N=1 gives the parabolic density profile:

$$\rho[x,y] = \frac{2\left(1 - \frac{x^2}{\mathrm{sx}^2} - \frac{y^2}{\mathrm{sy}^2}\right)}{\pi\ \mathrm{sx\ sy}}$$

The integral V[Z[t]] over *t* is performed giving the expression: $4\pi V[x,y] =$

$$\frac{4x^2y^2}{\mathrm{sxsy}(\mathrm{sx}+\mathrm{sy})^2} - \frac{4(\mathrm{sy}x^2 + \mathrm{sx}y^2)}{\mathrm{sxsy}(\mathrm{sx}+\mathrm{sy})} + \frac{2\left(\frac{(2\mathrm{sx}+\mathrm{sy})x^4}{\mathrm{sx}^3} + \frac{(\mathrm{sx}+2\mathrm{sy})y^4}{\mathrm{sy}^3}\right)}{3(\mathrm{sx}+\mathrm{sy})^2}$$

Forming the gradient, gives the field component Ex =

$$\left\{\frac{2x}{\pi\mathrm{sx}(\mathrm{sx}+\mathrm{sy})}\right\}\left\{1 - \frac{(2\mathrm{sx}+\mathrm{sy})x^2}{3\mathrm{sx}^2(\mathrm{sx}+\mathrm{sy})} - \frac{y^2}{\mathrm{sy}(\mathrm{sx}+\mathrm{sy})}\right\}$$

and a similar expression for component Ey with x and y, and sx and sy, interchanged. Fig. 1 shows the deviation from the linear rise of Ex in the x,y plane when sx = 2sy. Forming the Laplacian recovers the charge density.

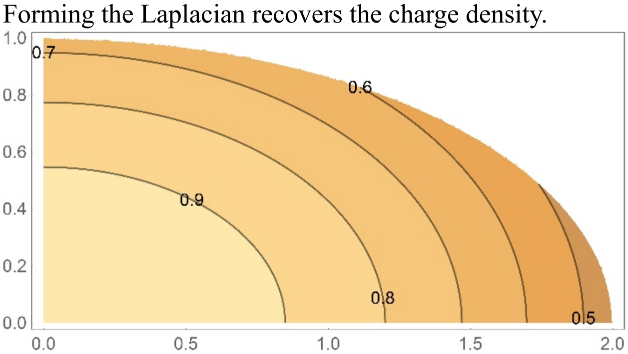

Figure 1: Electric field slope Ex(x,y)/x for N=1.

AUTOMATION

Performing these calculations by hand is tedious and error prone. Therefore, an automation strategy is needed. We re-derive the results of the previous section to illustrate a simple strategy for the analytic integration.

For N=1, the function to be integrated is $(1\text{-}Z)^2/S$. Let Z = (ax + ay) and expand the binomial form $(1\text{-}Z)^2$ -1.

Arrange the powers of ax,ay in an array:

$$\begin{pmatrix} 0 & -2\mathrm{ay} & \mathrm{ay}^2 \\ -2\mathrm{ax} & 2\mathrm{ax\ ay} & 0 \\ \mathrm{ax}^2 & 0 & 0 \end{pmatrix}$$

Now make the substitutions

$$\mathrm{ax} \to x\text{^}2/(\mathrm{sx}\text{^}2 + t), \mathrm{ay} \to y\text{^}2/(\mathrm{sy}\text{^}2 + t)$$

divide by S[t], and perform the integrations element by element. The sum of all the elements in the array is the desired potential:

$$\begin{pmatrix} 0 & -\frac{4y^2}{\mathrm{sxsy}+\mathrm{sy}^2} & \frac{2(\mathrm{sx}+2\mathrm{sy})y^4}{3\mathrm{sy}^3(\mathrm{sx}+\mathrm{sy})^2} \\ -\frac{4x^2}{\mathrm{sx}^2+\mathrm{sxsy}} & \frac{4x^2y^2}{\mathrm{sxsy}(\mathrm{sx}+\mathrm{sy})^2} & 0 \\ \frac{2(2\mathrm{sx}+\mathrm{sy})x^4}{3\mathrm{sx}^3(\mathrm{sx}+\mathrm{sy})^2} & 0 & 0 \end{pmatrix}$$

2D QUARTIC DENSITY, N=2

The procedure just described is now applied to the quartic density distribution

$$\rho[x,y] = \frac{3\left(1 - \frac{x^2}{\mathrm{sx}^2} - \frac{y^2}{\mathrm{sy}^2}\right)^2}{\pi\ \mathrm{sx\ sy}}$$

Expanding the binomial $[1\text{- (ax+ay)}]^3$ gives the coefficient array:

$$\begin{pmatrix} 0 & -3\mathrm{ay} & 3\mathrm{ay}^2 & -\mathrm{ay}^3 \\ -3\mathrm{ax} & 6\mathrm{axay} & -3\mathrm{axay}^2 & 0 \\ 3\mathrm{ax}^2 & -3\mathrm{ax}^2\mathrm{ay} & 0 & 0 \\ -\mathrm{ax}^3 & 0 & 0 & 0 \end{pmatrix}$$

Substituting for ax,ay, dividing by S[t] and integrating over t gives:

$$\begin{pmatrix} 0 & -\frac{6y^2}{\mathrm{sx\ sy}+\mathrm{sy}^2} & \frac{2(\mathrm{sx}+2\mathrm{sy})y^4}{\mathrm{sy}^3(\mathrm{sx}+\mathrm{sy})^2} & -\frac{2(3\mathrm{sx}^2+9\mathrm{sx\ sy}+8\mathrm{sy}^2)y^6}{15\mathrm{sy}^5(\mathrm{sx}+\mathrm{sy})^3} \\ -\frac{6x^2}{\mathrm{sx}^2+\mathrm{sx\ sy}} & \frac{12x^2y^2}{\mathrm{sx\ sy}(\mathrm{sx}+\mathrm{sy})^2} & -\frac{2(\mathrm{sx}+3\mathrm{sy})x^2y^4}{\mathrm{sx\ sy}^3(\mathrm{sx}+\mathrm{sy})^3} & 0 \\ \frac{2(2\mathrm{sx}+\mathrm{sy})x^4}{\mathrm{sx}^3(\mathrm{sx}+\mathrm{sy})^2} & -\frac{2(3\mathrm{sx}+\mathrm{sy})x^4y^2}{\mathrm{sx}^3\mathrm{sy}(\mathrm{sx}+\mathrm{sy})^3} & 0 & 0 \\ -\frac{2(8\mathrm{sx}^2+9\mathrm{sx\ sy}+3\mathrm{sy}^2)x^6}{15\mathrm{sx}^5(\mathrm{sx}+\mathrm{sy})^3} & 0 & 0 & 0 \end{pmatrix}$$

The sum of the array elements is the potential 4πV[x,y]. Forming the gradient, gives the field component Ex as the product of linear and quartic parts:

$$\frac{3x}{\pi\sigma_x(\sigma_x+\sigma_y)} \times \{1 + \frac{y^4(\sigma_x+3\sigma_y)}{3\sigma_y^3(\sigma_x+\sigma_y)^2} - \frac{2y^2(-3x^2\sigma_x + 3\sigma_x^3 - x^2\sigma_y + 3\sigma_x^2\sigma_y)}{3\sigma_x^2\sigma_y(\sigma_x+\sigma_y)^2} - \frac{x^2(-8x^2\sigma_x^2 + 20\sigma_x^4 - 9x^2\sigma_x\sigma_y + 30\sigma_x^3\sigma_y - 3x^2\sigma_y^2 + 10\sigma_x^2\sigma_y^2)}{15\sigma_x^4(\sigma_x+\sigma_y)^2}\}$$

And, likewise, for Ey. Fig. 2 shows the deviation from the linear rise of Ex in the x,y plane when sx = 2sy.

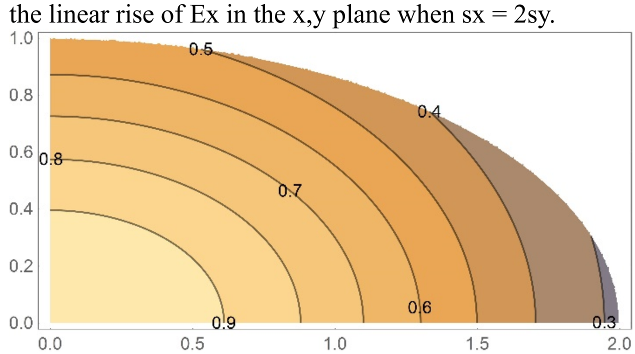

Figure 2: Electric field slope Ex(x,y)/x for N=2.

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2D DENSITY, N=6

$$\rho[x,y] = \frac{7(1-\frac{x^2}{\sigma_x^2}-\frac{y^2}{\sigma_y^2})^6}{\pi\sigma_x\sigma_y}$$

We apply the procedure. The leading terms are always linear; in this case:

$$\{\mathrm{Ex} \to \frac{7x}{\pi\sigma_x(\sigma_x+\sigma_y)}, \mathrm{Ey} \to \frac{7y}{\pi\sigma_y(\sigma_x+\sigma_y)}\}$$

The expressions for the field are too lengthy to report. Instead we plot Ex/x and Ey/y in the x,y plane in Fig. 3 and Fig. 4, respectively.

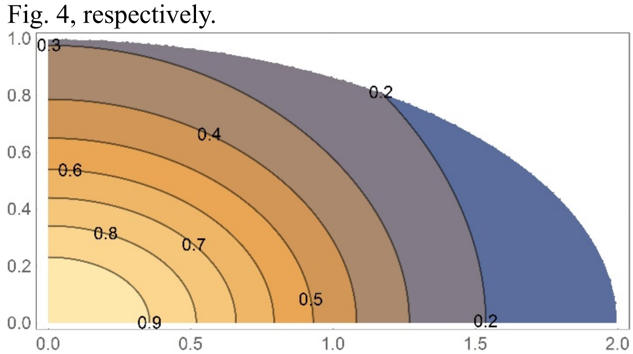

Figure 3: Electric field slope Ex(x,y)/x for N=6.

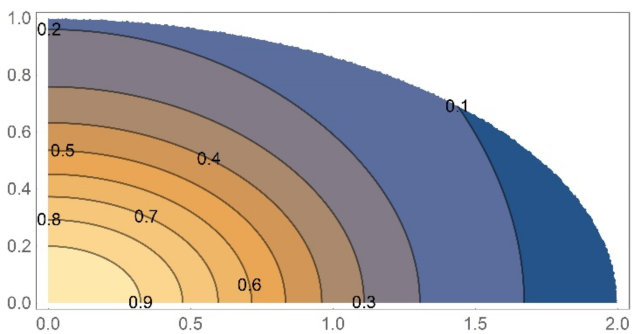

Figure 4: Electric field slope Ey(x,y)/y for N=6.

2D DENSITY, N=10

Already for N=6, the particle beam begins to develop a core and tails, like a gaussian distribution; and likewise for the electric field. This effect becomes more pronounced for increasing N; for example N=10, see Fig. 5 and Fig. 6.

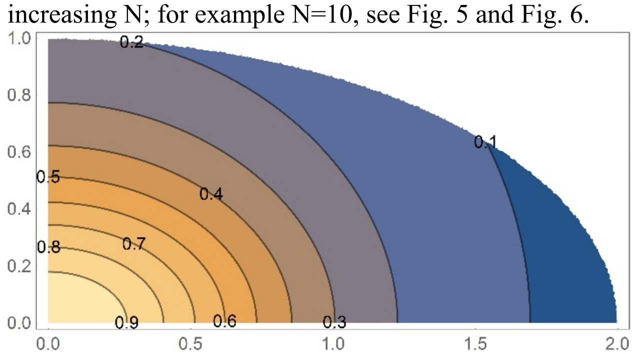

Figure 5: Electric field slope Ex(x,y)/x for N=10.

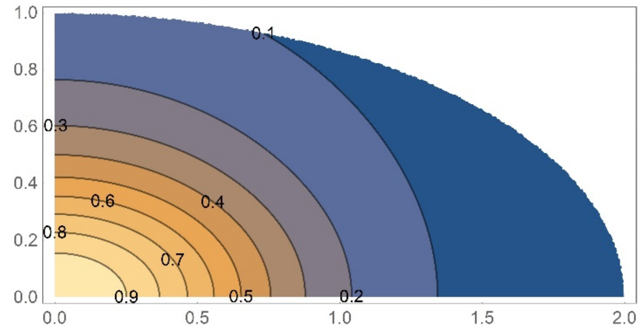

Figure 6: Electric field slope Ey(x,y)/y for N=10.

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CONCLUSION

We have developed a strategy for analytic calculation of the space-charge potential of elliptic cylinder hard-edge power-law binomial (charged particle) beams, normalized to constant r.m.s size. And we have presented a few examples. We have also given an historical narrative for this type of calculation.

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