

# NEW ANALYTICAL DERIVATION OF GROUP VELOCITY IN TW ACCELERATING STRUCTURES

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## Abstract

Ultra high-gradient accelerating structures are needed for the next generation of compact light sources. In the framework of the Compact Light XLS project, we are studying a high harmonic traveling-wave accelerating structure operating at a frequency of 35.982 GHz, in order to linearize the longitudinal space phase. In this paper, we propose a new analytical approach for the estimation of the group velocity in the structure and we compare it with numerical electromagnetic simulations that are carried out by using the code HFSS in the frequency domain.

## INTRODUCTION

The next generation of linear accelerators require unprecedented accelerating gradients for high energy physics and compact light sources. One of the main limitations to achieve ultra-high gradients, today around 100 MV/m [1], is the RF breakdown rate (BDR) which is defined as the number of breakdowns in the structure per unit time and length. Recently, an electromagnetic quantity called the modified Poynting vector has been demonstrated to be the main predictor for the BDR. It is defined as  $S_c = \text{Re}[S] + 1/6 \text{Im}[S]$  [2], where  $S$  is the Poynting vector describing the RF power flow through the traveling wave accelerating structure. As a consequence, the BDR is strictly related to the group velocity  $v_g$  which is proportional to  $S$  [3,4]. Therefore, the group velocity represents a crucial parameter to be characterized for each accelerating cavity [5]. We propose here an innovative analytical approach to the estimation of the group velocity which can be used before extensive 3D simulations. Moreover, the scaling laws of the group velocity with frequency and cavity dimensions are derived analytically, allowing to make an initial practical and useful choice of the main cavity parameters in order to design a structure to linearize the phase space for the CompactLight XLS project [6].

## GROUP VELOCITY DERIVATION

We apply the Bethe's theory [7] to a traveling wave accelerating structure. In Fig. 1 we show one cell with on-axis coupling through a circular aperture (iris). This theory states that the aperture is equivalent to electric or/and magnetic dipole moments. These dipole moments are proportional to the normal electric and tangential magnetic fields of the incident wave, respectively. We assume that the iris is small compared with the wavelength. It can be demonstrated that there is no magnetic dipole moment for an array of identical

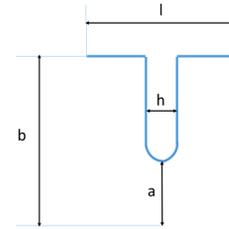


Figure 1: TW cavity shape for the  $2\pi/3$  mode.  $b$ ,  $a$ ,  $h$  and  $l$  are cavity radius, iris aperture radius, iris thickness and  $1/3$  of the cell length, respectively.

$TM_{01}$ -mode iris-loaded cavities and that the electric-dipole moment can be approximated as  $P = -2a^3 \epsilon_0 E_0 / 3$ , where “ $a$ ” is the aperture radius (see Fig. 1), and  $E_0$  is the unperturbed electric field. This electric dipole moment causes a perturbation and the result of this perturbation is the shift of the resonant frequency due to the interaction energy of the dipoles which changes the stored energy of the cavities. From Slater perturbation theorem and using the definition of group velocity [8],

$$v_g = \frac{d\omega}{dk_z} = K_1 c \left(\frac{a}{\lambda}\right)^3 \sin(\psi) e^{-\alpha h} \quad (1)$$

By considering the operating wavelength of the  $TM_{01}$  mode ( $\lambda = 2.61 b$ , where  $b$  is the cavity radius) we obtain,

$$v_g = K_2 c \left(\frac{a}{b}\right)^3 \sin(\psi) e^{-\alpha h} \quad (2)$$

where  $\psi$  is the phase advance per cell,  $h$  is the iris thickness, and  $K_1$  and  $K_2$ , depending on the operating mode and the cavity geometry, are equal to 33.72 and 1.895, respectively. Replacing  $\alpha \approx \frac{2.405}{a}$  expanding the exponential function with a Taylor series and considering  $h = 0.08\lambda$  for a practical periodic accelerating structures, Eqs. (1) and (2) can be written as,

$$\frac{v_g}{c} = K_1 \sin(\psi) \left[ \left(\frac{a}{\lambda}\right)^3 - 0.19 \left(\frac{a}{\lambda}\right)^2 + 0.0185 \left(\frac{a}{\lambda}\right) - 0.0012 + 0.000057 \left(\frac{\lambda}{a}\right) \right] \quad (3)$$

$$\frac{v_g}{c} = K_2 \sin(\psi) \left[ \left(\frac{a}{b}\right)^3 - 0.502 \left(\frac{a}{b}\right)^2 + 0.126 \left(\frac{a}{b}\right) - 0.0211 + 0.0026 \left(\frac{b}{a}\right) \right] \quad (4)$$

We observe that the above equations are independent of the operating frequency and lead to a fast and accurate way

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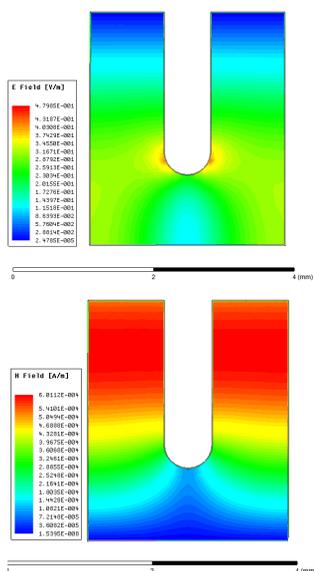


Figure 2: Electric and magnetic field magnitudes for the  $TM_{01}$  mode of the high accelerating periodic structure. Minimum value of the electric and maximum value of the magnetic fields are near the outer surface of the cavity.

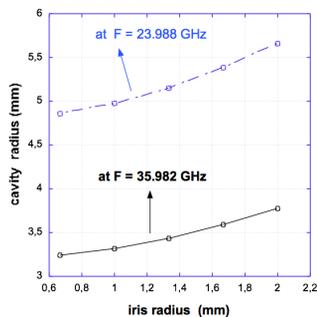


Figure 3: Cavity radius as a function of the iris radius at 23.988 GHz and 35.982 GHz.

to estimate the group velocity as a function of the geometry of the structure only. The advantage of analytical approach is that it doesn't require heavy computation which is of course time and resource intensive while by numerical methods the group velocity per each TW cell is calculated after few iterations in order to find the operating frequency of the desired mode (for a fixed iris radius  $a$ , it usually means adjusting the cell radius  $b$ ). Numerical simulations become extensive especially when long TW structures are designed for constant-gradient operation. Indeed, in such case, each single cell has a different iris radius and therefore different cavity radius for which the group velocity needs to be calculated each time.

## SIMULATION RESULTS

In this section we discuss the group velocity obtained by "HFSS code" in ANSYS [9]. The RF power is fed to the periodic structure and the electromagnetic mode is excited with  $120^\circ$  phase advance per cell. By applying proper

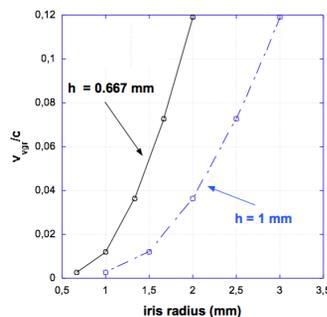


Figure 4: Group velocity ( $v_g/c$ ) as a function of the iris radius at 23.988 GHz (blue-dashed line) and 35.982 GHz (black-solid line).

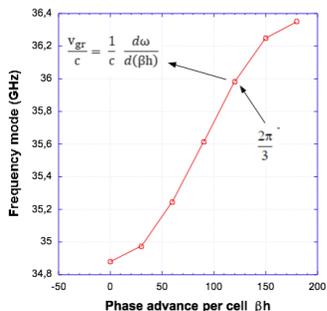


Figure 5: Frequency mode as a function of the phase advance of the TW structure for 35.982 GHz. The iris radius, iris thickness, cavity radius are 1.3333 mm, 0.6667 and 3.4345 mm, respectively.

boundary conditions, it can be avoided to simulate the entire structure because the code HFSS allows to simulate periodic structures using only one cell, as shown in Fig. 1.

Electric and magnetic field distributions are shown in Fig. 2. The minimum value of the electric and the maximum value of the magnetic fields are near the outer surface of the cavity as they are expected to be for the  $TM_{01}$  mode. Additional simulations in order to estimate the cavity radius and the group velocity as a function of the iris radius for different operating velocity (second and third harmonics of the X band) are also shown in Figs. 3 and 4. By increasing the iris radius, the cavity radius is also increased in order to keep constant the operating frequency, as we can observe from Fig. 3. When the iris radius raises, also the resonant frequency increases and we should enlarge the cavity radius to maintain the cavity in a resonant mode at the same frequency. Figure 4 shows the variation of group velocity with respect to the iris radius. The group velocity increases at higher frequencies.

Figure 5 shows the dispersion curve of the cell which represents the frequency modes as function of the phase advance per cell in the TW structure. In the following equation, in order to calculate the group velocity, we use the slope of the curve (i.e. frequency shift per phase shift):

$$\frac{v_g}{c} = \frac{2\pi h}{c} \frac{df}{d\phi} \quad (5)$$

where  $\phi$  is the phase advance per cell. Assuming a structure with the operating frequency  $f=35.982$  GHz, the iris

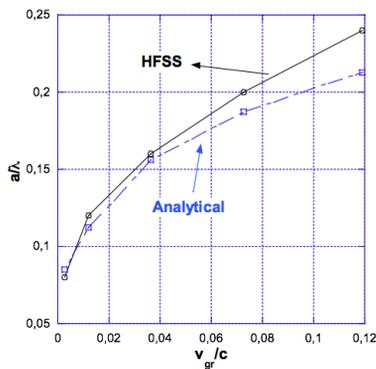


Figure 6: The ratio  $a/\lambda$  as a function of the group velocity  $v_g/c$ . Comparison between the HFSS and Analytical estimations. The iris radius and RF wavelength are  $a$  and  $\lambda$ , respectively.

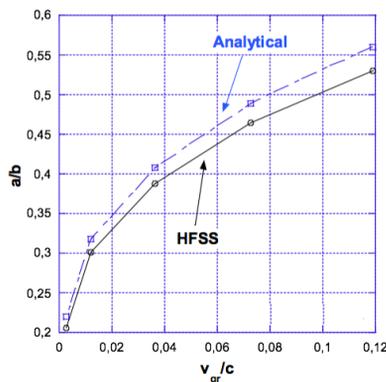


Figure 7: The ratio  $a/b$  as a function of the group velocity  $v_g/c$ . Comparison between the HFSS and Analytical estimations. The iris radius and cavity radius are  $a$  and  $b$ , respectively.

radius, iris thickness and cavity radius are given by 1.333 mm, 0.667 and 3.434 mm, respectively. The corresponding group velocity for this structure is 0.0365  $c$ .

## COMPARISON BETWEEN HFSS SIMULATION AND ANALYTICAL RESULTS

In this section we compare the analytical estimation of the group velocity as a function of the iris radius with the numerical results. From Fig. 4 we observe that the group velocity increases with a steeper slope for higher frequencies in agreement with Eq. (1). By keeping the same iris radius and varying the frequency, the group velocity is bigger for higher frequencies. Figures 6 and 7 show the behaviors of  $a/\lambda$  and  $a/b$  as a function of the group velocity independent from the operating frequency. Increasing the group velocity,  $a/\lambda$  and  $a/b$  increase, both analytically and numerically. As an example from Fig. 6 when  $\frac{a}{\lambda} = 0.160$  and the related wavelength of the operating frequency ( $f=35.982$  GHz) is  $\lambda=8.3375$  mm, then the group velocity and the corresponding iris radius are 0.0365 $c$  and 1.333 mm, respectively. The corresponding  $a/b$  for this group velocity is 0.388 as we can obtain from Fig. 7. Consequently the cavity radius would

be 3.434 mm which is the same cavity radius obtained from HFSS simulations. The relative error between analytical and numerical results for  $a/b$  as function of  $v_g/c$  is almost constant (5% of the simulation results). By increasing the iris radius we should increase cavity radius to maintain the resonant frequency, and  $a/b$  raises simultaneously with a constant ratio as the group velocity increases. We have a variation of the relative error for  $a/\lambda$  as function of  $v_g/c$  because we should keep constant the RF wavelength. Considering a constant value for  $\lambda$  and increasing the cavity radius,  $a/\lambda$  should be incremented with different ratio because numerically this ratio increases with a higher value respect to the analytical value.

It should be noted that increasing the iris radii, the errors between the analytical approach and HFSS are amplified in absolute values while they are constant in relative ones. One physical reason is that we used an electric polarization coefficient for electric moment which is  $\alpha = -2/3a^3$ . The assumption for using this coefficient is that the holes should be small compared with the wavelength. It is possible to extend the model to bigger size by considering a factor  $e^{ika}$  in the normal electric field in which the variation in the Green's function must be considered and this correction can be of the order of  $(ka)^2$  rather than  $ka$  [7]. New ideas were investigated in order to solve this problem [10–12]. The author of [12] solved the problem of diffraction of arbitrary electromagnetic field by a circular perfectly conducting disk using a series representation in powers of  $k$  using the results of generalized Babinet's principle [13] and considering that the disk problem and the aperture problem are equivalent. Taking two terms instead of the whole equation we have,  $P_z = \frac{4}{3}a^3\epsilon_0 E_z^0(1 - \frac{3}{10}(ka)^2)$ . The change in the stored energy would be less than that we obtained from electric-dipole moment based on Bethe's theory. In this case, the energy change causes a smaller perturbation, which leads to less frequency shift and consequently gives a smaller group velocity variation, bringing our prediction and simulations results closer.

## CONCLUSION

In this paper, we applied the Bethe's theory to the circular aperture of coupled cavities that can be approximated as an electric dipole for the  $TM_{01}$  mode and we observed that the perturbation due to the interaction energy of these dipoles leads to the variation of stored energy.

We then demonstrated that the group velocity can be obtained from these variations using different polarization coefficients. We compared the analytical and numerical results and we have observed that group velocity shows a good agreement when the holes are small compared with the wavelength applying the electric dipole moment obtained with the Bethe's theory. Furthermore, when the irises are comparable in size with the wavelength, we suggested to use an electric dipole moment considering the variation of the electric field in the Green's function adding a term of the order  $(ka)^2$  as a correction factor.

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