

STUDY OF INTEGRABLE AND QUASI-INTEGRABLE SEXTUPOLE LATTICE*

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Abstract

In order to maximize beam lifetime in circular particle accelerators, the nonlinear beam optics are optimized to maximize the dynamic aperture of the beam. The dynamic aperture (DA), which is a 6-D phase space volume of stable trajectories, depends on the strength of the nonlinearities in the machine, and is calculated via particle tracking. Current DA optimization processes include multi-objective genetic algorithm optimizers, and relies on minimizing the magnitudes of resonance driving terms (RDT), which are calculated from the nonlinear contribution to the one-turn-map. The process of searching through the parameter space for an ideal combination that maximizes DA is computationally strenuous. By setting up the sextupole channel such that it resembles a symplectic integrator of a smooth Hamiltonian, with only a few sextupoles we are able to closely reproduce phase space trajectories of a smooth Hamiltonian up to the hyperbolic point. No chaos and resonances are observed if phase advance per one sextupole magnet in the channel does not exceed $0.12 \times 2\pi$. Therefore, an important property of the suggested approach is the intrinsic elimination of the resonances, and minimization of corresponding RDTs.

INTRODUCTION

In this study we focus on methods for eliminating the 3rd order harmonic resonance which is excited by the use of sextupole magnets due to their $\propto x^3$ potential. The phase space of an accelerator which uses many sextupole magnets is therefore not only prone to the 3rd order harmonic resonance, but also higher order resonances which may appear within the volume of allowed trajectories. This volume is determined by the position of the hyperbolic point, which creates a separatrix beyond which all trajectories diverge to infinity.

While the purpose of using sextupoles in accelerator lattices is primarily to correct chromaticity, here we explore the way in which the third order resonance can be eliminated in a 1-dimensional toy model, without regard to chromaticity compensation. This is an exercise based on the findings of Danilov and Nagaitsev [1], in which they showed how to achieve a time independent Hamiltonian in normalized coordinates in the limit of a smooth distribution of nonlinear magnets.

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THEORY

As shown in Danilov and Nagaitsev (DN) [1], a transformation exists which can result in a time-dependent integrable Hamiltonian for a simple accelerator lattice consisting of drifts, thin sextupoles, and a focusing element. The full derivation, shown in [1], reveals that appropriately scaling the magnet strength distribution by a power of the β function through the channel will result in resonance elimination. In the case of sextupoles, once the transformation to normalized coordinates:

$$x_N = \frac{x_N}{\sqrt{\beta_x}}, \quad p_{x,N} = p\sqrt{\beta_x} + \frac{\beta_x' x}{2\sqrt{\beta_x}}, \quad (1)$$

is performed, the effective Hamiltonian is then:

$$H_N = \frac{p_{x,N}^2}{2} + \frac{x_N^2}{2} + \beta_x(s)V(x_N\sqrt{\beta_x}, s). \quad (2)$$

Therefore, for a sextupole potential $V(x) \sim \frac{\alpha x^3}{3}$,

$$V(x_N) \propto \beta_x(s) \frac{(\sqrt{\beta(s)}x)^3}{3} K_3(s). \quad (3)$$

From here we can see that by choosing a sextupole distribution $K_3(s)$ to cancel out the s dependence in the potential:

$$K_3(s) \sim \frac{K_3^{(0)}}{\beta_x(s)^{\frac{5}{2}}}, \quad (4)$$

the desired time-independent $V(x_N)$ can be achieved.

From this theory, we can see that by distributing sextupoles in a drift space with their strength proportional to $\beta_x(s)^{-5/2}$, we can eliminate the time-dependence of the potential in the Hamiltonian, and create an invariant.

Beside the DN prescription, another suggestion for eliminating the 3rd order resonance comes from simply observing the nature of the 3rd order resonance driving term (RDT). The magnitude of the RDT associated with the 3rd order sextupole resonance is defined as [2]:

$$h_3 = \frac{-1}{24} \sum_{i=1}^N K_3(s_i) \beta(s_i)^{3/2} e^{i3\mu(s_i)} \quad (5)$$

From this expression, we became curious if modifying the DN solution of $K_3(s) \propto \beta_x(s)^{-5/2}$ to $K_3(s) \propto \beta_x(s)^{-3/2}$ to cancel out the β function dependence in the RDT would also help with resonance elimination. In order to achieve similar time-independence as in the DN solution, it was also determined that the sextupoles should be spaced such that the phase advance between sextupoles through the channel,

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$\mu(s_i)$, was constant. In contrast, the DN solution does not have spatial symmetry requirements (defined as an equidistant distribution of sextupoles) as described in the original work. Therefore, to demonstrate the DN solution we constructed the channel with the sextupoles placed $\frac{L}{N_{\text{sex}}}$, where N_{sex} is the number of sextupoles in the channel, as long as the strength of the magnets is determined by the $\beta_x(s)^{-5/2}$ proportionality.

To illustrate this better, the Figure 1 shows general schematics for these two toy model solutions.

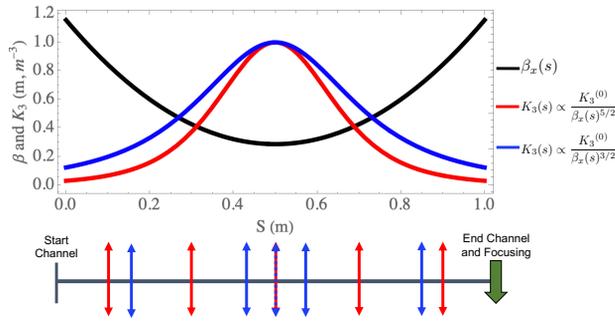


Figure 1: A schematic representation of the sextupole channel, where the β function and the two suggested sextupole strength distributions $K_3(s) \propto \beta_x(s)^{-5/2}$ and $K_3(s) \propto \beta_x(s)^{-3/2}$ are shown in red and blue respectively. Beneath the plot is a visualization of the sextupole positioning in real space along the length of the channel. The red double-ended arrows represent the spatially symmetric sextupole distribution, and the blue double-ended arrows represent the constant phase advance spacing; the colors also correspond to the magnetic strength distribution used.

1-D MATCHED OPTICS CHANNEL

Here, we demonstrate how the DN “matched-optics” prescription is constructed at tunes, ν , very close to $2\pi/3$. To demonstrate this, a drift-kick lattice was constructed in Mathematica [3] in the following way, to track particles in (x, p_x) phase space.

First, the channel consists of a drift space which can be modified to include the desired number of thin, zero-length, sextupoles. At the end of the channel is a thin element which provides linear focusing. While we consider only a 1-D example here, in the 2-D extension the beta functions are matched exactly and there is linear focusing provided in both transverse planes.

The beta function for a drift length L , and focusing k is given by:

$$\beta_x(s) = \frac{L - sk(L - s)}{\sqrt{1 - (1 - \frac{Lk}{2})^2}}. \quad (6)$$

The DN map is constructed in a drift-kick model, in which the drift map is the simple 1-D drift transfer map, and the focusing at the end of the channel is a standard focusing transformation. Details for these maps can be found in [4].

The linear focusing constant k is determined by the desired betatron phase advance and tune in the following way. The phase advance is calculated $\phi = (\frac{1}{3} + \delta)2\pi$ for detuning δ away from the $1/3$ resonance. The linear focusing is then calculated from the phase advance as:

$$k = -2 \frac{\cos(\nu) - 1}{L}. \quad (7)$$

The non-linear kick provided by a sextupole at position s_i along the channel is shown in Eq. 8, with the sextupole strength given by the function $K_3(s)$.

$$\begin{pmatrix} x_{i+1} \\ p_{x,i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ p_{x,i} + x_i^2 K_3(s_i) \end{pmatrix}. \quad (8)$$

A tracking code was written in Mathematica to transform a particle initialized in (x, p_x) phase space through the channel.

RESULTS AND DISCUSSION

Applying the DN solution directly, various channels were created and phase space portraits calculated, in order to determine the least number of sextupoles possible for resonance elimination. It was observed that the resonance was not eliminated as initially suggested.

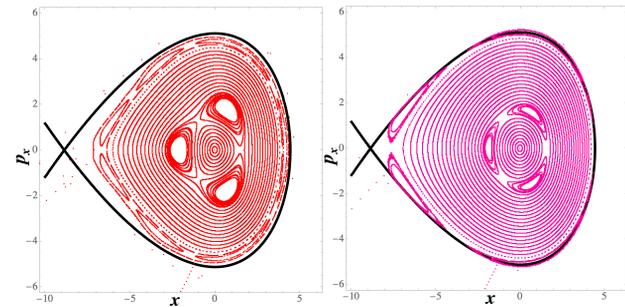


Figure 2: Phase space portraits for 1-dimensional particle tracking through a sextupole channel of unit length, for 5 thin sextupoles; (left) the DN solution, (right) the DN solution with RDT optimization for $K_3(s) \propto \beta_x(s)^{-\alpha}$, and $\alpha = 2.12$. The tune ν_x is set to be $1/3 + \delta$ for $\delta = 0.005$, which results in a total phase advance for the cell $\phi = 2.12$.

This can be seen clearly in Fig. 2, where large resonance islands are visible near the origin, and the region of closed orbits near the origin is small and slightly deformed due to the presence of the resonance. We applied RDT minimization to further reduce the magnitude of the resonance. This minimization was done in Mathematica by searching for the appropriate exponent α for $K_3(s) \propto \beta_x(s)^{-\alpha}$ which would further reduce the RDT of the system. The results of this RDT optimization are shown in Fig. 2. The resonance is still present, but its strength is reduced, as shown by the decrease in the size of the islands. Therefore, it is clear that there is yet an optimal configuration.

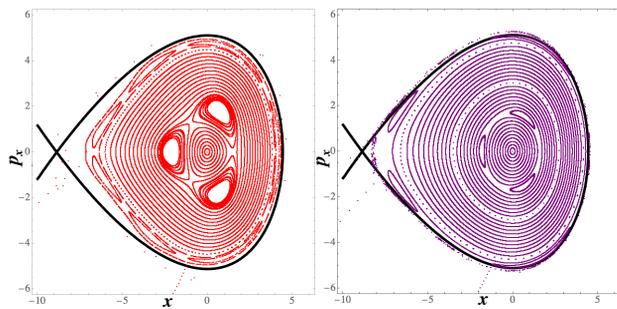


Figure 3: Phase space portraits for 1-dimensional particle tracking through a sextupole channel of unit length, for 5 thin sextupoles; (left) the DN solution with asymmetric sextupole spacing and, (right) the DN solution with restored spatial symmetry as shown in Fig. 1. The tune ν_x is set to be $1/3 + \delta$ for $\delta = 0.005$, which results in a total phase advance for the cell $\phi = 2.12$.

Due to the observation that $K_3(s)$ must be changed significantly in order to improve the dynamics, we tried adjusting the physical spacing of the sextupoles in the channel such that the periodicity is maintained with each cycle. While the sextupoles were placed with equidistant spacing, the symmetry was broken at the periodic point, where the distance was $2\frac{L}{N_{\text{sex}}}$ instead of $\frac{L}{N_{\text{sex}}}$. After restoring the spatial symmetry, but retaining $K_3(s) \propto \beta_x(s)^{-5/2}$, it was determined that the symmetry of the channel is vital for resonance mitigation. The results of this adjustment are shown in Fig. 3.

With further observation, and by exploring the behavior Eq. 5, and based on the determination that the spatial positions of the sextupoles did contribute to the resonance mitigation, it was clear that the phase advance between the sextupoles was an important parameter. Similarly, from Eq. 5, it is clear that the time-dependence (or s dependence) can be cancelled out by matching the power of the β function dependence, and setting $K_3(s) \propto \beta_x(s)^{-3/2}$. By calculating a sextupole mesh along the channel such that the phase advance between the sextupoles was constant, the following results were observed:

From the results shown in Fig. 4, it is clear that by maintaining periodicity by ensuring that the phase advance between sextupoles is constant and by applying a matching condition for $K_3(s)$, the resonance is entirely eliminated. For this cell with total phase advance $0.3383 \times 2\pi$, with only 3 sextupoles (thus phase advance between sextupoles is no more than $0.12 \times 2\pi$), the resonance is entirely eliminated.

Figure 5 shows that with the new prescriptions of $K_3(s) \propto \beta_x(s)^{-3/2}$, with only 3 sextupoles, the third order harmonic resonance is entirely eliminated.

CONCLUSIONS AND FUTURE WORK

The analysis shown is purely 1-dimensional, with no coupling between the horizontal and vertical coordinates. In a realistic system this cannot be achieved, though it can be

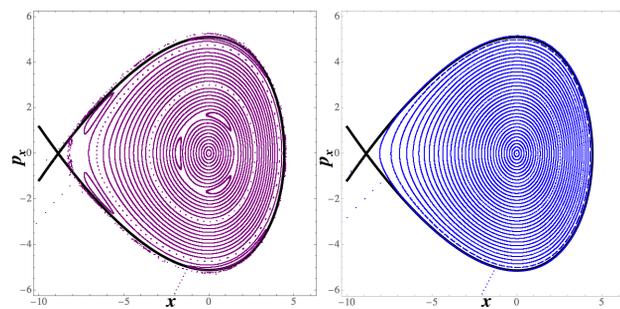


Figure 4: Phase space portraits for 1-dimensional particle tracking through a sextupole channel of unit length, for 5 thin sextupoles; (left) the DN solution with asymmetric sextupole spacing and, (right) the equal phase advance sextupole placement as shown in Fig. 1 and $K_3(s) \propto \beta_x(s)^{-3/2}$. The tune ν_x is set to be $1/3 + \delta$ for $\delta = 0.005$, which results in a total phase advance for the cell $\phi = 2.12$.

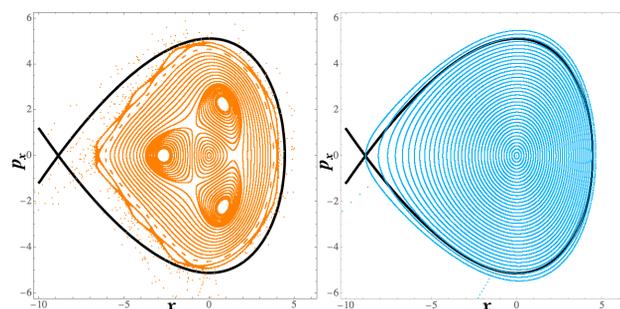


Figure 5: Phase space portraits for 1-dimensional particle tracking through a sextupole channel of unit length, (left) for 5 thin sextupoles with DN solution for asymmetric sextupole spacing and, (right) 3 thin sextupoles and the equal phase advance sextupole placement as shown in Fig. 1 and $K_3(s) \propto \beta_x(s)^{-3/2}$. The tune ν_x is set to be $1/3 + \delta$ for $\delta = 0.005$, which results in a total phase advance for the cell $\phi = 2.12$.

emulated by (unmatched beta functions). Therefore, while this method has been demonstrated using particle tracking in phase space, it is prudent to extend this analysis into 2-dimensions. The feasibility of such a system in practice can therefore be determined by constructing realistic lattice in simulation software in order to understand how robust this method is.

ACKNOWLEDGMENTS

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