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THE BETATRON EQUATION WITH THE SYNCHRO-BETATRON COUPLING TERM AND SUPRESSION OF THE COUPLED-BUNCH MODE

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Abstract

The synchrotron oscillation, which is both longitudinal and horizontal oscillations, occurs under a constant longitudinal velocity of revolving particle.The synchrotron and betatron equations for revolving particles are derived from the improved Hamiltonian. The betatron equation accompanys the shinchro-betatron resonant coupling term. The coherent synchrotron oscillation frequency of the bunch is defined from the integrated phase. Taking advantage of the resonant coupling term, an experiment to suppress magnetically the destabilized coupled-bunch mode of the synchrotron oscillation is proposed.

INTRODUCTION

We discussed the oscillating synchrotron motion [1] using the Hamiltonian composed of coasting, synchrotron and betatron motions. We call it the synchrotron oscillation. The synchrotron oscillation is not only the longitudinal oscillation of the revolution frequency but also the horizontal oscillation of the average radius. It is unveiled that the longitudinal velocity v of revolving particle is constant under the synchrotron oscillation. On the frame of revolving particle, the synchrotron oscillation is a pure horizontal oscillation. Therefore, if we can somehow artificially suppress the horizontal oscillation, we can suppress the synchrotron oscillation as a whole. The Hamiltonian, which clarified the synchro-betatron resonant coupling mechanism in a storage ring, revealed that the energy exchange between the betatron and synchrotron oscillations was possible ($\bar{x}$ and δ_s are coupled) since both oscillations have the horizontal component [2]. However, the synchro-betatron resonant coupling term did not arise naturally. The synchrotron oscillation coupled with the betatron oscillation around the on-momentum (reference) closed orbit but not with that around the off-momentum closed orbit. In this manuscript, a scale factor term of the geometry $(\bar{x}/\rho)\delta_s$ is introduced into the Hamiltonian to overcome this problem. Then the synchrotron oscillation will be derived and discussed related to the coherent synchrotron mode of the bunch. We propose a method to suppress the destabilized coupled-bunch mode of the synchrotron oscillation through the synchro-betatron resonant coupling term.

THE IMPROVED HAMILTONIAN FOR REVOLVING PARTICLES

In the right-handed curvilinear coordinate $(x, s, z$, x is the horizontal coordinate, p_x is the horizontal momentum and s is the orbital length. We assume that an on-momentum particle of mass m and momentum p_0 is revolving on the orbit of the average radius R under the dipole magnetic field $-B$. δ is the (rationalized) fractional deviation, ϕ is the phase and t is time. Then the orbit angle is $\theta = s/R$, the revolution frequency is $\omega = d\theta/dt$ and the (longitudinal) velocity is $v = ds/dt$, which satisfies $v = \beta c$ and c is the velocity of light. For the off-momentum closed orbit, $\bar{x}$ is the horizontal coordinate, $\bar{p}_x$ is the horizontal momentum, ϕ_D is the phase delay, D is the dispersion function and ρ is the radius of curvature. They are defined in [1]. The prime denotes differentiation with respect to s. Keeping up to the 2nd order to describe a revolving particle with coasting, betatron and synchrotron motions, the Hamiltonian $\bar{H}$ composed of three motions is obtained [2]. Now we neglect the DC component δ_C of the fractional deviation $\delta = \delta_C + \delta_s$ for convenience ($\delta \rightarrow \delta_s$). The oscillating component δ_s is the fractional deviation of the kinetic energy caused by the synchrotron oscillation. Then, add a scale factor term $(\bar{x}/\rho)\delta_s$ in the coasting motion. The improved Hamiltonian turns to be

$$H = -(1+\delta_s) + (\bar{x}/\rho)\delta_s + \frac{1}{2}\left(\frac{\bar{p}_x}{p_0}\right)^2 + \frac{1}{2}K_s\bar{x}^2 + \frac{1}{2}(-\eta)\delta_s^2 \tag{1}$$
$$- \frac{hqV}{2\pi\beta^2 E_0}[\cos(\phi+\phi_D) - \cos(\phi_s+\phi_D) + (\phi-\phi_s)\sin(\phi_s+\phi_D)]$$

where the coasting motion consists of the 0th (on-momentum particle) and 1st (δ_s) order effects. In the 0th order, $\phi = \omega t$. ϕ_s is the synchronous phase. h is the harmonic number and $h\omega$ is the RF frequency. V is the (effective) RF voltage. η is the phase slip factor. Since v is a constant in the synchrotron oscillation [1], we have $s = \int v\, dt = v(t-t')$ where t' is an arbitrary reference time. Then

$$\theta = \frac{s}{R} = \frac{v(t-t')}{R}\,. \tag{2}$$

Now it is possible to replace $s-$description with $t-$description in the next section.

We assume $\phi \rightarrow \phi_s$ for the synchrotron oscillation in the following discussion.

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THE BETATRON EQUATION WITH THE SYNCHRO-BETATRON COUPLING TERM

Hamilton's equations of motion for $(\bar{x}, \bar{p}_x/p_0)$ are

$$\frac{d\bar{x}}{ds} = \frac{\partial H}{\partial(\bar{p}_x/p_0)} = \frac{\bar{p}_x}{p_0} \tag{3}$$

$$\frac{d(\bar{p}_x/p_0)}{ds} = -\frac{\partial H}{\partial \bar{x}} = -K_x\bar{x} - \frac{\delta_s}{\rho} \tag{4}$$

From Eqs. (3) and (4), the betatron equation of motion with the synchro-betatron resonant coupling is obtained

$$\frac{d^2\bar{x}}{ds^2} + K_s\bar{x} = -\frac{\delta_s}{\rho} \tag{5}$$

Since the polarity of δ_s is chosen arbirarily, this equation corresponds to Eq. (33) of [2]. The synchrobetatron resonant coupling term arises naturally. Now the betatron motion is coupled with the synchrotron motion around the off-momentum closed orbit ($\bar{x}$ and δ_s are coupled).

THE SYNCHROTRON EQUATION AND THE COHERENT SYNCHROTRON OSCILLATION FREQUENCY

Hamilton's equations of motion for (ϕ, δ_s) are obtained from H as

$$\frac{d\phi}{d\theta} = \frac{\partial H}{\partial \delta_s} = -1 + \frac{\bar{x}}{\rho} + (-\eta)\delta_s - \left[D'\left(\frac{\bar{p}_s}{p_0}\right) + DK_x\bar{x}\right]. \tag{6}$$

We have

$$\frac{d}{ds}\left[D'\left(\frac{\bar{p}_x}{p_0}\right) + DK_x\bar{x}\right] = \frac{1}{\rho}\frac{d\bar{x}}{ds} - \frac{dD}{ds}\frac{\delta_s}{\rho}.$$

Since particles generally revolve more than 10^4 times for one oscillation, it is possible to consider that δ_s is a constant during one revolution. Then,

$$D'\left(\frac{\bar{p}_x}{p_0}\right) + dK_x\bar{x} = \frac{\bar{x}}{\rho} - \frac{\delta_s}{\rho}\int_C D'\, ds + C_1 = \frac{\bar{x}}{\rho} + C_1 \tag{7}$$

where $D' = dD/ds$ and $\int_C dD = \int_C D'ds = 0$ for the circumference C in the closed orbit of the circular ring. C_1 is an integraion constant. From Eq. (6),

$$\frac{d\phi}{d\theta} = \frac{\partial H}{\partial \delta_s} = -1 + (-\eta)\delta_s - C_1 . \tag{8}$$

We consider a small amplitude oscillation of δ_s. Putting $\phi \to \phi_s$, we obtain

$$\frac{d\delta_s}{d\theta} = -\frac{hqV\ \cos(\phi_s + \phi_D)}{2\pi\beta^2\ E_0}(\phi - \phi_s) \tag{9}$$

and

$$\frac{d^2\delta_s}{d\theta^2} = -\frac{hqV\ \cos(\phi_s + \phi_D)}{2\pi\beta^2\ E_0}\frac{d\phi}{d\theta} = -\frac{hqV\ \cos(\phi_s + \phi_D)}{2\pi\beta^2\ E_0}[(-\eta)\delta_s - (1 + C_1)] . \tag{10}$$

This is the synchrotron equation of motion

$$\frac{d^2}{d\theta^2}(\delta_s - \delta_0) = -\nu_s^2(\delta_s - \delta_0) \tag{11}$$

where $\delta_0 = (1 + C_1)/(-\eta)$ and

$$\nu_s^2 = \frac{\omega_s^2}{\omega^2} = \frac{hqV\ |\eta\cos(\phi_s + \phi_D)|}{2\pi\beta^2\ E_0} .$$

ν_s is the synchrotron tune, ω_s is the angular synchrotron frequency, and θ_0 is an arbitrary orbit angle. From Eq. (11) the synchrotron equation of motion is obtained, using Eq. (2):

$$\delta_s = \hat{\delta}\cos[\nu_s(\theta - \theta_0)] + (1 + C_1)/(-\eta) \tag{12}$$
$$\to \hat{\delta}\cos[\omega_s(t - t_0)] + (1 + C_1)/(-\eta)$$

where t_0 is an arbitrary reference time which satisfies $t_0 = t' + \theta_0/\omega$. Since ν is a constant, an oscillation in θ-coordinate is converted into an oscillation in t-coordinate. We have the following relation [3]:

$$\frac{\Delta\omega}{\omega} = -\eta\delta_s , \tag{13}$$

where $\Delta\omega$ is the deviation of angular revolution frequency. From Eq. (12)

$$\frac{\Delta\omega}{\omega} = \frac{\Delta\hat{\omega}}{\omega}\cos[\omega_s(t - t_0)] + 1 + C_1 , \tag{14}$$

where $\Delta\hat{\omega}$ is the oscillation amplitude which satisfies $\Delta\hat{\omega}/\omega = -\eta\hat{\delta}$. From Equations (8), (13), and (14), the 1st order variation of ϕ is

$$d\phi = \frac{\Delta\hat{\omega}}{\omega}\cos[\omega_s(t - t_0)]\ d\theta . \tag{15}$$

Then, including the 0th order term,

$$\frac{d\phi}{dt} = \Delta\hat{\omega}\cos[\omega_s(t - t_0)] + \omega . \tag{16}$$

The synchrotron oscillation frequency is observed as sidebands of ω. If we average $\Delta\hat{\omega}$ for the particles in a bunch to see its coherent effect, the time variation of averaged ϕ is in tune with the RF frequency because of the phase stability principle near RF phase 0 or π. It is observed as sidebands of the RF frequency $h\omega$ (not as sidebands of ω). These spectra are classified as the coherent synchrotron mode of the bunch [4]. Assume $\Delta\hat{\omega}$ is a function of ξ and define $\bar{\phi}$ as the averaged value of ϕ. From Eq. (16),

$$\frac{d\bar{\phi}}{dt}\int \Delta\hat{\omega}(\xi)\rho(\xi)d\xi\ \cos[\omega_s(t - t_0)] , \tag{17}$$

where $\rho(\xi)$ is a distribution function of a bunch, normalized by $\int\rho(\xi)d\xi = 1$, with a proper parameter ξ defined around the center of gravity of the bunch. Taking one period average, we can define the coherent synchrotron oscillation frequency $\omega_{\rm cs}$ as follows:

$$\omega_{\rm cs} = \int_{\rm average} d\bar{\phi} = \int \Delta\hat{\omega}(\xi)\rho(\xi)d\xi \int_{\rm average}\cos(\omega_s(t - t_0)]\ dt = \frac{1}{2}\int \Delta\hat{\omega}(\xi)\rho(\xi)d\xi . \tag{18}$$

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THE DESTABILIZED COUPLED-BUNCH MODE AND ITS SUPRESSION

The ion beam was bunched by the RF frequency of 2.5192 MHz ($h = 100$) in S-LSR, Kyoto University [2]. Sidebands of synchrotron frequencies of the resonant tune ($\bar{x}$ and δ_s are coupled) were observed at both sides of each harmonics around the center peaks of RF frequency (See the Figure 6 of [5]). The synchrotron oscillation amplitude in the right side sideband of $h = 99$ harmonics is prolonged. The synchrotron oscillation amplitude in the left side sideband of $h = 101$ harmonics is also prolonged (See Figure 9 and 10 of [5]). These frequencies are located near the RF frequency. These prolonged side bands represent the coupled-bunch mode, in which the sidebands of each harmonics flanked the RF center peak. The energy of the RF wave is somehow fed into those nearby frequencies in the resonant tune when $\bar{x}$ and δ_s are coupled. Tune jump (ω_s has an uncertainty) was observed near the resonant tune in S-LSR experiment [2].

In the resonant tune, those sideband frequencies may be influenced by a strong RF wave of their neighbour. The prolonged side bands develop and are destabilized as the beam current increases in the J-PARC Main Ring operation, which is a high intensity proton synchrotron of 30 GeV [6]. The coupled-bunch mode was destabilized as the beam current increases. A longitudinal mode-by-mode feedback system was installed to stabilize it. An extra RF cavity was used as a longitudinal kicker [7]. However, it may not work well as the beam space charge increases since an electrostatic suppressing procedure is used.

Let's consider the time development of Eq. (12), which is given by integrating δ_s with time as follows,

$$\int \delta_s(t)\, dt = \hat{\delta} \int \cos\left[\omega_s(t - t_0)\right]\, dt + (1 + C_1)t/(-\eta)\,. \tag{19}$$

The 1st term of RHS is oscillating. For an arbitrary particle, we can choose $C_1 = -1$ in the 2nd term, which contribute nothing. However, it turns to be $C_1 \to -1 - (\delta_s/\rho)\int_C D' ds \neq -1$ if K_x in Eq. (7) is changed by the beam space charge

$$D'\left(\frac{\bar{p}_x}{p_0}\right) + DK_x\bar{x} = \frac{\bar{x}}{\rho} - \frac{\delta_s}{\rho}\int_C D' ds - 1\,.$$

Then $\hat{x}$ and δ_s are again coupled: the energy of the RF wave is somehow fed into those nearby frequencies. The 2nd term, accordingly δ_s, slowly increases and δ_s is destabilized. This is the destabilized coupled-bunch mode. To suppress it, δ_s should be contained directly. In fact the inhomogeneous Hill's equation has the term of the magnetic imperfection [8]. Adding the imperfection term together, Eq. (5) turns to be

$$\frac{d^2\bar{x}}{ds^2} + K(s)\bar{x} = \frac{\Delta B_Z(t)}{B\rho} - \frac{\delta_s(t)}{\rho} \tag{20}$$

where $\Delta B_Z(t)$ is an additional oscillating dipole field. For a very slow time scale compared to the betatron oscillation, we can neglect LHS of Eq. (20). If an additional oscillating dipole field is supplied so that $\Delta B_Z(t)/B = \delta_s(t)$ is satisfied in the case $\delta_s(t)$ represents the particular destabilizes coupled-bunch mode, $\delta_s(t)$ will be suppressed. We propose an experiment to suppress the coupled-bunch mode with an additional oscillating dipole field $\Delta B_Z(t)$. Detailed experimental arrangements will be discussed in future publication.

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