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PONDEROMOTIVE INSTABILITY OF SELF-EXCITED CAVITY*

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Abstract

The electro-magnetic (EM) fields within a super-conducting radio frequency (SRF) cavity can be sufficiently strong to deform the cavity shape, which may lead to a ponderomotive instability. Stability criteria for the self-excited mode of cavity operation were given in 1978 by Delayen. The treatment was based on the Routh-Hurwitz analysis of the characteristic polynomial. With the Wolfram modern analytical tool, 'Mathematica', we revisit the criteria for an SRF cavity equipped with amplitude and phase loops and a single microphonic mechanical mode.

INTRODUCTION

Whereas generator driven RF cavity systems have been used for charged-particle acceleration for nearly a century, self-excited (SE) resonance has been considered [1,2] for only three decades. SE has two-parameters (Θ,Ψ) and is less intuitive. Our starting point is the masterful exposition by Delayen [2]. It must be emphasized that SE loop is an enabling technology for SRF. The EM resonance width is exceedingly small compared with the excitation frequency; so, without prior knowledge, finding (and driving) the resonance can be difficult until its location is known. And, of course, Lorentz force detuning (LFD) will change the resonant frequency as the amplitude is increased. A numerical treatment is given by Joshi [3].

Basics

The SE loop is essentially a narrow band resonator equipped with positive feedback. The loop contains the resonator, a near-linear amplifier, an adjustable phase shifter, and a limiter and attenuator to control the amplitude.
The resonator has loaded quality factor and time constant Q_c and τ, respectively. The loop phase is initially adjusted to be $2n\pi$ at the resonance frequency ω_c with n integer. The shifter then introduces an addition phase Θ_L. The loop responds by oscillating at the SE frequency ω, given by:

$$2\mathrm{Tan}[\Theta_L]\omega[t] = -\tau(\omega_c^2 - \omega[t]^2)$$

Here it is assumed that ω_c has already the static LFD included and compensated.

In contra-distinction to generator driven (GD), it is important to understand that Θ_L is the "cause" and ω is the "effect". In SE mode, the excitation amplitude is self-stabilized. Following Delayen, we begin by considering the stability of the SE oscillator with no control loops. Let v[t] and $v_g[t]$ be the cavity voltage and equivalent generator voltage. They are governed by:

$$\omega_c^2 v[t] + \frac{2v'[t]}{\tau} + v''[t] = \frac{2v_g'[t]}{\tau}$$

where primes denote time derivatives.

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We write the voltages in the following forms:

$$\{v = e^{i\Phi[t]}V[t], v_g = e^{i\Theta_L + i\Phi[t]}V_g[t], \omega[t] = \Phi'[t]\}$$

with the steady state (denoted subscript 0) conditions:

$$V_{g0} = \mathrm{Sec}[\Theta_L]V_0$$

We now introduce deviations from the steady state,

$$\{V[t] \to \mathrm{V0} + \delta\mathrm{V}[t], V_g[t] \to \mathrm{Vg0} + \delta\mathrm{Vg}[t]\}$$

$$\{\omega_c^2 \to (\delta\omega\mu + \omega_c)^2, \omega[t] \to \delta\mathrm{w}[t] + \omega[t]\}$$

where δωμ is dynamic LFD.

We suppose the EM resonator to be coupled to a mechanical mode (of the RF cavity) having quality factor Q and resonance frequency Ω. This mode gives a static LFD $\Delta\omega_\mu = -k_\mu V_0^2 < 0$. The normalized (dimensionless) coupling constant is $K_L = 2\tau k_\mu V_0^2 > 0$.

We linearize the equations of motion, and take the Laplace transform w.r.t. frequency-like variable *s*. We introduce the vector $\boldsymbol{u} = \{a_v, \delta\omega, a_g, \delta\omega\mu\}$ where a_v and a_g are amplitude modulation indices. The system matrix is **P** =

$$\begin{pmatrix} 1 + s\,\tau_c & \frac{s\,\tau_c}{2\,\omega} & -1 & 0 \\ -\frac{s}{\omega} - \frac{s^2\,\tau_c}{2\,\omega} + \mathrm{Tan}[\Theta_L] & \tau_c & -\mathrm{Tan}[\Theta_L] & -\tau_c \\ 0 & 0 & 1 & 0 \\ \frac{K_L}{\tau_c} & 0 & 0 & 1 + \frac{s^2}{\Omega^2} + \frac{s}{Q\,\Omega} \end{pmatrix}$$

and the condition **P.u**=**0** leads to the characteristic determinant and polynomial in *s*. Delayen discards the term in Tan[Θ]/ω as being small. This is not self-consistent, because in the following we shall see that Tan[Θ] may be as large as $4Q_c$ which is in principle very large for an SRF cavity. Nevertheless, we set Tan[Θ]/ω=0.

Depending on precisely which terms in *s* we retain, the polynomial may be a monomial, cubic, quartic or quantic. We present the conditions arising from each of these choices. All the terms $\{-s/\omega, -(s\text{^}2\,\tau)/2\omega, s\tau/2\omega\}$ are small; if they are all neglected, then the coupling to the mechanical mode and to Tan[Θ] both disappear leading to a damped cavity response $1 + s\tau = 0$. If we retain only the small term $-s/\omega$, column 1 row 2, the result is the same.

Cubic

If we retain only the small term $s\tau/(2\omega)$, row 1 col 2, the result is a cubic $a_0 + sa_1 + s^2a_2 + s^3a_3$. The term a_0 does not contain K_L or Tan[Θ], so there is no monotonic instability. $\{a_1, a_2, a_3\}$ all contain Tan[Θ], but only a_1 contains K_L. Sufficient conditions for all coefficients a_i>0 and Routh determinants RH_j >0 are Tan[Θ] < $4Q_c$ and $K_L < \frac{2\omega}{Q\Omega}$ and K_L << $4Q_c$.

Quartic

If we retain only the two small terms $\{-s/\omega, s\tau/2\omega\}$ the result is a quartic $a_0 + sa_1 + s^2a_2 + s^3a_3 + s^4a_4$. $\{a_0, a_4\}$

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do not contain K_L or Tan[Θ], so there is no monotonic instability. $\{a_1, a_2, a_3\}$ all contain Tan[Θ], but only a_1 contains K_L. Sufficient condition for all a_i>0 is Tan[Θ] < $4Q_c$.

Sufficient condition for all RH_j>0 is $K_L < \frac{\omega}{Q\Omega}$. Alternatively, Tan[Θ] < $3Q_c$ and $K_L < \frac{2\omega}{Q\Omega}$ is sufficient. Generally:

$$\{2Q^2\omega\Omega K_L\} < 8\omega\Omega Q_c + 4Q(\omega^2 + 4\Omega^2 Q_c^2) - 2\Omega(\omega + 4Q\Omega Q_c)\text{Tan}[\Theta_L] + Q\Omega^2\text{Tan}[\Theta_L]^2$$

Quintic

Retaining all small terms leads to a quintic. This case is treated by Delayen. The coefficients are:

$$\{a_0 = 4Q\omega^2\Omega^2, a_4 = 2Q + \tau\Omega, a_5 = Q\tau\}$$

$$a_1 = 2\omega\Omega(2\omega(1 + Q\tau\Omega) - Q\Omega K_L - Q\Omega\text{Tan}[\Theta_L])$$

$$a_2 = 4\tau\omega^2\Omega + 2Q(2\omega^2 + \Omega^2) - 2\omega\Omega\text{Tan}[\Theta_L]$$

$$a_3 = 2\Omega + Q\tau(4\omega^2 + \Omega^2) - 2Q\omega\text{Tan}[\Theta_L]$$

Sufficient condition for all a_i>0 is Tan[Θ] < $4Q_c$.

When Tan[Θ] =0, RH_3 & RH_4 >0 automatically, leaving RH_5 to determine stability. Delayen gives

$$K_L < 2\tau\omega + \frac{2\omega}{Q\Omega} - \frac{2Q\tau\omega}{Q + \tau\Omega} = \frac{2\omega}{Q\Omega} + 4Q_c - \frac{4Q\omega Q_c}{Q\omega + 2\Omega Q_c}$$

More accurately, we find:

$$K_L < 2\tau\omega + \frac{2\omega}{Q\Omega} - \frac{2Q\tau\omega(4Q + \tau\Omega)}{(2Q + \tau\Omega)^2} = \frac{2\omega}{Q\Omega} + 4Q_c - \frac{2Q\omega Q_c(2Q\omega + \Omega Q_c)}{(Q\omega + \Omega Q_c)^2}$$

The expressions for limiting K_L agree to leading order.

Consider now non-zero loop phase, $\Theta_L \neq 0$. Sufficient condition for RH_3 & RH_4 >0 is Tan[Θ] < $2Q_c$. The fifth Routh determinant, RH_5, is the most challenging. When Tan[Θ] < $2Q_c$, a sufficient condition is $K_L < Q_c$. This corresponds to a very large static LFD of $\Delta\omega_\mu = -\frac{\omega}{4}$.

Two points are noted: (i) in contra-distinction to GD, the microphonic does not un-couple when Tan[Θ]=0; (ii) a_0 does not contain K_L so there is no monotonic instability.

The general conclusion is that SE-oscillator without control loops will not encounter a ponderomotive instability. Moreover, the stability limits that derive from the small terms $\{-s/\omega, -(s\text{^}2\,\tau)/2\omega, s\tau/2\omega\}$ are so far away that we may as well neglect them all, and recover the matrix **P**=

$$\begin{pmatrix} 1 + s\tau & 0 & -1 & 0 \\ \text{Tan}[\Theta_L] & \tau & -\text{Tan}[\Theta_L] & -\tau \\ 0 & 0 & 1 & 0 \\ \frac{K_L}{\tau} & 0 & 0 & 1 + \frac{s^2}{\Omega^2} + \frac{s}{Q\Omega} \end{pmatrix}$$

PHASE & AMPLITUDE LOCK

We must lock our SE oscillator to an external reference for the frequency and amplitude. Following Delayen, the loop is modified to include quadrature control, B[t]. The equivalent generator voltage becomes:

$$v_g = e^{i\Theta_L + i\Phi[\tau]}(1 + iB[\tau])V_g[\tau]$$

The dynamical equations for the resonator become:

$$\tau\omega_c^2 V[\tau] - \tau V[\tau]\omega[\tau]^2 = = -2B[\tau]\text{Cos}[\Theta_L]\omega[\tau]V_g[\tau] - 2\text{Sin}[\Theta_L]\omega[\tau]V_g[\tau]$$

$$V[\tau]\omega[\tau] + \tau\omega[\tau]V'[\tau] = = \text{Cos}[\Theta_L]\omega[\tau]V_g[\tau] - B[\tau]\text{Sin}[\Theta_L]\omega[\tau]V_g[\tau]$$

Here ω[t] is the loop frequency when B is present. In the steady state it is equal to the reference frequency defined by $2\text{Tan}[\Psi]\omega[\tau] = +\tau(\omega_c^2 - \omega[\tau]^2)$. Note the sign is reversed compared with Delayen; we chose the convention to agree with the generator driven case following Schulze [4]. We introduce the static values:

$B_0 = -\text{Tan}[\Theta + \Psi]$ and $V_{g0} = \text{Cos}[\Psi + \Theta_L]\text{Sec}[\Psi]V_0$

We then consider small perturbations in the dynamical variables, linearize about the steady state, and Laplace transform. There is a new state vector $\boldsymbol{u} = \{a_v, \delta\omega, a_g, \delta B, \delta\omega\mu\}$ and system matrix **P** =

$$\begin{pmatrix} 1 + s\,t & 0 & -1 & \text{Cos}[\Theta + \Psi]\,\text{Sec}[\Psi]\,\text{Sin}[\Theta] & 0 \\ -\text{Tan}[\Psi] & t & \text{Tan}[\Psi] & -\text{Cos}[\Theta]\,\text{Cos}[\Theta + \Psi]\,\text{Sec}[\Psi] & -t \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{KL}{t} & 0 & 0 & 0 & 1 + \frac{s^2}{w0^2} + \frac{s}{Q\,w0} \end{pmatrix}$$

High Gain Phase Loop

The SE oscillator with a high gain phase loop, locked to an external frequency is the analogue of the GD case. For simplicity, we take the feedback to be a perfect integrator of the frequency deviation. The matrix elements P[row,col] =P[4,2]=P[4,5]=F/s where F>0 is constant.

In the absence of the microphonic (K_L = 0), the characteristic is quadratic. Examination of the coefficients a_l show the conditions $\{\Theta \to -\Psi, \Theta \to \Psi, \Theta \to \pi/2 - \Psi\}$ to be good, poor, and disastrous, respectively.

When K_L >0, there is a quartic in *s*. For example, the DC term: $a_0 = FQ\Omega^2\text{Cos}[\Psi + \Theta_L]\text{Sec}[\Psi]$ $(\text{Cos}[\Theta_L] + \text{Sin}[\Theta_L](2K_L - \text{Tan}[\Psi]))$

In the regime of interest, $\{\text{Cos}[\Theta + \Psi] > 0, \text{Cos}[\Theta] > 0\}$, but this still leaves four combinations:

- Both below resonance $\{\text{Tan}[\Psi] > 0, \text{Sin}[\Theta] < 0\}$
- Both above resonance $\{\text{Tan}[\Psi] < 0, \text{Sin}[\Theta] > 0\}$
- One low, one high $\{\text{Tan}[\Psi] > 0, \text{Sin}[\Theta] > 0\}$
- One high, one low $\{\text{Tan}[\Psi] < 0, \text{Sin}[\Theta] < 0\}$.

Low/low gives the monotonic instability. High/high gives the oscillatory instability. The mixed cases may give instabilities also. For simplicity and brevity, we present only the low/low and high/high cases; but experimentalists beware the mixed cases!

Monotonic instability (low/low) From the coefficient a_0>0, we find the threshold:

$$2K_L < (-\text{Cot}[\Theta_L] + \text{Tan}[\Psi])$$

Substituting $\Theta \to -\Psi$, yields the GD threshold:

$$K_L < \text{Csc}[2\Psi]$$

All other a_i>0 automatically.

Oscillatory instability (high/high) All Routh determinants except RH_4 are greater than zero. RH_4>0 is challenging to analyse. RH_4 is linear in K_L, so we can write $RH_4 = k_0 + k_1 \times K_L$ where k_j are functions of F and the EM and MM resonator parameters. K_L is then the quotient $-k_0/k_1$. We expand this in inverse powers of F>>1.

Let $\rho = \tau\Omega$. The threshold leading terms are:

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$$K_L < \frac{(Q+\rho+Q\rho^2)\mathrm{Cot}[\Theta_L]}{2Q^2\rho} - \frac{(2Q+\rho)\mathrm{Tan}[\Psi]}{2Q^2\rho} + \frac{\mathrm{Tan}[\Psi]^2\mathrm{Tan}[\Theta_L]}{2Q\rho}$$

(Delayen gives a similar expression, but has the wrong sign for the term linear in Tan[Ψ].) The next to leading order terms are:

$$\frac{(Q+\rho+Q\rho^2)\mathrm{Cos}[\Psi]\mathrm{Csc}[\Theta_L]\mathrm{Sec}[\Psi+\Theta_L]}{2FQ^3} + \frac{(-Q-\rho+2Q^2\rho)\mathrm{Sec}[\Theta_L]\mathrm{Sec}[\Psi+\Theta_L]\mathrm{Sin}[\Psi]}{2FQ^3}$$

The special case $\Theta \rightarrow -\Psi$ can be treated exactly.

$$\{-2Q\rho(Q+FQ+\rho)^2K_L\mathrm{Tan}[\Psi]\} < (Q+\rho+Q\rho^2)(F^2Q+F\rho+Q\rho^2) + F(FQ(Q+\rho)+\rho(Q+\rho-2Q^2\rho)+FQ^2\mathrm{Sec}[\Psi]^2)\mathrm{Tan}[\Psi]^2$$

PHASE & AMPLITUDE LOOPS

For simplicity, we take the amplitude feedback to be pure proportional to a_v. The matrix elements P[4,2]=P[4,5] =F/s and P[row,col] =P[3,1] =A where A>0 is constant. This results in a quartic characteristic equation.

Tan[Θ_L] =0

Let us point out immediately that setting Θ_L identically zero, has the effect that all coefficients a_i and all Routh determinants RH_j are automatically greater than zero provided A, F, Q, $\rho=\tau\Omega$ all >0. In such case B_0= -Tan[Ψ]. In this special, but important, case K_L and Tan[Ψ] are absent from all a_i & RH_j. (This happens because we omitted the small couplings $\{-s/\omega, -(s\hat{}2\,\tau)/2\omega, s\tau/2\omega\}$.) So Θ_L=0 is the ideal regime; but inevitably there are phase and/or detuning errors, so we move toward the general case.

Tan[Θ_L] +Tan[Ψ] =0

The next most simple case is $\Theta+\Psi$=0, or B_0=0. This condition means that the setpoint for the feedback is zero, but whatever signal arrives it must be added in quadrature.

In this case, we need consider the stability only as a function of Ψ, which is related to the difference of reference and SE-oscillation frequencies.

Monotonic condition The term a_0 may change sign when Tan[Ψ]>0, leading to the threshold: $0 < K_L < (1+A)\mathrm{Csc}[2\Psi]$. So amplitude feedback has a significant beneficial effect for operation below resonance.

Oscillatory condition All other a_i and RH_j are automatically >0, except for

RH_4=

$$(A+F)(Ap+(A^2+p^2)Q)(Fp+(F^2+p^2)Q) + F\mathrm{Tan}[\Psi](2Q\rho((A+F)Q+\rho)^2K_L + A(A+F)((A+F)Q\rho+\rho^2+Q^2(AF-2\rho^2) + AFQ^2\mathrm{Sec}[\Psi]^2)\mathrm{Tan}[\Psi])$$

which may become negative when Tan[Ψ]<0. Here, for brevity, A stands in place of (A+1).

General Case

There are two parameters (Θ ,Ψ) leading to four combinations: low-low, high-high, low-high, high-low as above. It simplifies matters to stipulate $\mathrm{Cos}[\Theta+\Psi]\mathrm{Sec}[\Psi] > 0$, so that cavity and generator V and V_g have the same sign. First we find conditions for a_i>0:

	Low-low	High-high	Low-high	High-low
a_0	×	√	mixed	mixed
a_1	√	√	CotΘ>TanΨ	CotΘ<TanΨ
a_2	√	√	CotΘ>TanΨ	CotΘ<TanΨ
a_3	√	√	√	√

Monotonic condition In particular, below resonance, $\mathrm{Tan}[\Psi] > 0\&\ \mathrm{Tan}[\Theta] < 0$ we find the threshold condition:

$$-2K_L < (1+A)\mathrm{Cos}[\Psi+\Theta_L]\mathrm{Csc}[\Theta_L]\mathrm{Sec}[\Psi]$$

But generally it is more complicated, see Fig.1, which shows also the mixed cases.

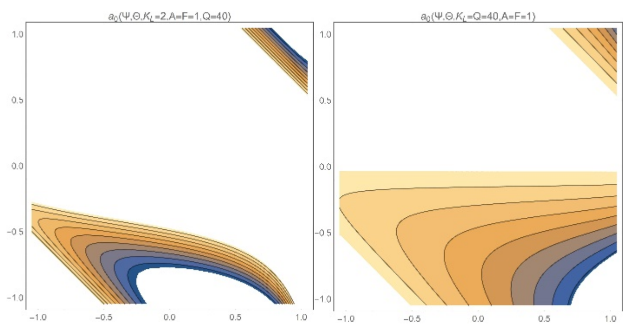

Figure 1: regions a_0>0 shown white, a_0 <0 coloured. Abscissa Ψ, ordinate Θ. Left/right = low/high Lorentz coupling. The classical monotonic regime is the lower right quadrant.

Oscillatory condition Now we consider the Routh determinants, only above and below resonance: RH_3>0 always, but RH_4 may change sign and the parametric behaviour is complicated. The asymptotic expansion (1/F ⟶0) used above does not give simple results when A>0, because the felicitous cancellations do not occur. The working is lengthy and reveals that the meaning of "very large gains" is F≥Q^2 and A≥Q^2 in order to cover the range of ρ=[1, Q].

Although *Mathematica*® can calculate RH_4 exactly, to obtain an expression short enough for this paper we must introduce some approximations. In the region |Θ|≤ π/4 and |Ψ|≤π/4, $\mathrm{Cos}[\Psi+\Theta_L]\mathrm{Sec}[\Psi]$ has average value 0.9, so we replace the matrix elements as P[1,4]=Tan[Θ] and P[2,4] = -1. RH_4 >0 yields the upper limit on LFD detuning:

$$\{2Fp Q(p+(A+F)Q)^2K_L\} < (A+F)((A\rho+Q(A^2+\rho^2))(F\rho+Q(F^2+\rho^2))\mathrm{Cot}[\Theta_L] + AF\mathrm{Tan}[\Psi](-(A+F)Q\rho-\rho^2+2Q^2(-AF+\rho^2) + AFQ^2\mathrm{Tan}[\Psi]\mathrm{Tan}[\Theta_L]))$$

CONCLUSION

Following Delayen, we have rederived, corrected, and extended the criteria for avoiding ponderomotive instabilities for a self-excited cavity operating with phase and amplitude loops. The criteria are rather similar to those of a

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generator driven RF cavity, particularly when Θ+Ψ=0. We draw attention to the mixed cases where simplistic tuning above or below resonance may be insufficient for stability.

REFERENCES

[1] D. Gorle, D. Leroy, P. Morignot: "Operation Of An Auto-oscillator Utilizing A Superconducting Cavity In The Reaction Loop", CERN-MPS-MU-SD-70-1, June 1970.

[2] J.R. Delayen: "Phase and Amplitude Stabilization of Superconducting Resonators", Doctoral Thesis, California Institute of Technology, 1978.

[3] G. Joshi, V. Agarwal, G. Kumar, and R. G. Pillay, "Development of the Model of a Self Excited Loop", in *Proc. 23rd Particle Accelerator Conf. (PAC'09)*, Vancouver, Canada, May 2009, paper WE3RAC06, pp. 1886-1888

[4] D. Schulze: "Ponderomotive Stability of RF Resonators and Resonator Control Systems", ANL-TRANS-944, 1971.

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