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PONDEROMOTIVE INSTABILITY OF GENERATOR-DRIVEN CAVITY*

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Abstract

The electro-magnetic (EM) fields within a super-conducting radio frequency (SRF) cavity can be sufficiently strong to deform the cavity shape, which may lead to a ponderomotive instability. Stability criteria for the generator-driven mode of cavity operation were given in 1971 by Schulze. The treatment side-stepped the Routh-Hurwitz analysis of the characteristic polynomial. With the Wolfram modern analytical tool, 'Mathematica', we revisit the criteria for an SRF cavity equipped with amplitude, phase, & tuning loops and a single microphonic mechanical mode.

INTRODUCTION

The fundamental EM mode of the RF cavity is coupled to mechanical modes (MM) of the cavity via Lorentz force: charges and currents on the interior surfaces of the cavity are acted upon by the electric and magnetic fields at those surfaces to produce what is called a *ponderomotive* force. At low field, as occurs in normal conducting cavities, this effect is negligible. In SRF cavities the fields may be so high as to initiate an electro-mechanical instability. This effect was noted in the 1970's, leading to the analysis of Schulze[1], and there has been little analytical study since that time. Such is the mathematical complexity, that most researchers rely on numerical simulations of particular cases, and Schulze relied on a mix of analytic and heuristic arguments based on the Nyquist criterion, rather than explicit use of the Routh-Hurwitz stability criteria for the roots of the characteristic polynomial. Herein, we use the modern symbolic mathematics tool 'Mathematica', to manipulate the very lengthy mathematical formulae.

METHOD

In the neighbourhood of an isolated cavity resonance, the mode can be modelled by an LCR resonator; quantified by the resonance angular frequency ω_c, loaded quality factor Q_c, and drive frequency ω. As noted by Schulze, the cavity mechanical modes (MM) can each be represented by their normal coordinate qμ, angular frequency Ωμ and quality factor Qμ and coupling Fμ to the EM mode; which satisfy

$$\ddot{q}_\mu + \frac{2}{\tau_\mu}\dot{q}_\mu + \Omega_\mu^2 q_\mu = \frac{\Omega_\mu^2}{c_\mu} F_\mu$$

In the case that the cavity and fundamental EM mode has cylindrical symmetry, and the cavity tuner does not break this symmetry, then only the longitudinal MMs couple to the fundamental. Typically, only a few MMs lie in the range 0-300 Hz; and for analysis we focus on one alone.

Cavity Resonance with Lorentz Force Detuning

The Lorentz force detuning (LFD) is the sum over all MMs, each contributing proportional to the square of the field (E). Schulze implies that the net detuning is always negative: $\omega_c(|E|>0) < \omega_c(E=0)$, and that contributions from individual MMs are also negative.

At low field we may sweep the drive frequency to map out the impedance Z_c=RCosΨ Exp[iΨ] where the phase angle is Tan[Ψ]=$(\omega_c^2-\omega^2)/(2\alpha\omega)$ and $\alpha=\omega_c/(2Q_c)$.

For a driven oscillator, Ψ is considered a response to ω.

At high field, the Lorentz detuning leads to a distorted amplitude and phase response versus drive frequency‡. To recover the simple form Z_c it is assumed that the static LFD detuning is exactly compensated by the cavity tuner.

We can then linearize the equations of motion for the EM mode and the MM mode, for small perturbations.

Let $\Delta\omega_c$(DC)= $-kV_0^2$ and $\delta\omega_c$(AC)= $-kV_0^2 a_v$. Let $-\delta\omega_c\tau = K_L a_v$ then $-K_L=\Delta\omega_c\tau$ is dimensionless coupling strength.

Cavity Modulation Response

The cavity response to modulations of the generator current and resonance frequency is given by Koscielniak [2]. The modulation indices *a,p* are dimensionless. Subscripts *g,v* denote "generator" and cavity voltage, respectively.

Routh-hurwitz (RH) Stability Analysis

After forming the Laplace transform of the dynamical equations, there results a characteristic polynomial; the roots of which determine the stability of the system. Given

$$s^5 a_0 + s^4 a_1 + s^3 a_2 + s^2 a_3 + s a_4 + a_5$$

the determinants RH_j generate constraints on the coefficients a_i >0 such that all roots have negative real part. This does not exclude oscillations, but they are damped.

Dynamical Equations

The amplitude, phase and tuning loop gains are Ka(s), Kp(s), Kt(s). The product of system matrix **P**

$$\begin{pmatrix} 1+s\tau & \mathrm{Tan}[\Psi] & -1 & -\mathrm{Tan}[\Psi] & 0 & 0 \\ -\mathrm{Tan}[\Psi] & 1+s\tau & \mathrm{Tan}[\Psi] & -1 & -1 & 0 \\ \mathrm{Ka} & 0 & 1 & 0 & 0 & 0 \\ 0 & \mathrm{Kp} & 0 & 1 & 0 & 0 \\ 0 & \mathrm{Kt} & 0 & -\mathrm{Kt} & 1 & -\frac{2\Omega^2 K_L}{\mathrm{M2}} \\ \frac{\mathrm{M2}}{s^2+\frac{s\Omega}{Q}+\Omega^2} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and vector **v**= $\{av, pv, ag, pg, \delta\omega\tau, q\}$ equals zero. It might be thought that the DC response of the MM should be

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‡ It is noteworthy that the "jump condition" in the non-linear case, and the small-amplitude stability criterion, both formulated by Schulze, are identical. This is because the distortion of the resonance curve can be removed by a linear transformation: it is simply a shear.

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omitted, since it is compensated by the tuner, but the AC modulations have no DC component, so it is unnecessary.

With no deliberative detuning, i.e. Ψ=0, the MM decouples from the EM mode. Moreover, when Ψ=0, the loops decouple from one another, and stability is assured for pole-zero-cancellation or PID style control.

NO LOOPS

When loop gains are zero, the polynomial is a quartic with coefficients $a_0 = Q\Omega^2(\mathrm{Sec}[\Psi]^2 - 2K_L\mathrm{Tan}[\Psi])$, $a_1 = \Omega(2Q\tau\Omega + \mathrm{Sec}[\Psi]^2)$, $a_2 = \tau\Omega(2 + Q\tau\Omega) + Q\mathrm{Sec}[\Psi]^2$, $a_3 = \tau(2Q + \tau\Omega)$ and $a_4 = Q\tau^2$. a_1, a_2, a_3 are automatically greater than zero. $a_0>0$ for Ψ<0 or limited Lorentz strength $K_L\mathrm{Sin}[2\Psi]<1$ and Ψ>0. Violating these conditions leads to the *monotonic* instability. The minimum occurs at Ψ=π/4; so $K_L<1$ is sufficient for stability. Let $\rho = \tau\Omega$. All the Routh determinants are automatically positive, except RH_4 which may change sign when Ψ<0 leading to the limitation:

$$-K_L Q\rho(2Q+\rho)^2 < (Q+\rho+Q\rho^2)^2\mathrm{Cot}[\Psi]$$
$$+(Q^2+2Q\rho+\rho^2-2Q^2\rho^2+Q^2\mathrm{Sec}[\Psi]^2)\mathrm{Tan}[\Psi]$$

This is virtually identical to the condition for *oscillatory* instability inferred by Schulze under the assumption

$$\frac{\Omega}{2\omega_c}\left\{\frac{1}{\tau\Omega}+\tau\Omega\right\} \ll 1$$

This assumption is removed; parameters may be chosen freely. A lower bound on K_L results from setting Ψ= -π/4:

$$\mathrm{KL} < \frac{2\rho^2+2Q\rho(2+\rho^2)+Q^2(4+\rho^4)}{Q\rho(2Q+\rho)^2}$$

The threshold is sensitive to the choice of ρ and Q.

Classical Regime

$\rho \geq Q$, the cavity bandwidth (BW) is much less than the mechanical mode frequency. In this case, Fig.1, threshold K_L is much larger than for the monotonic; and stability is usually obtained by setting Ψ<0.

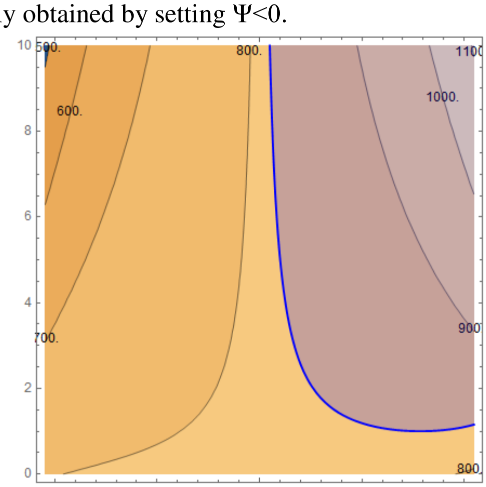

Figure 1: Criteria a_0 and RH4, in the space of Ψ and K_L.

For $\rho = Q$ & large mechanical Q, RH_4 is approximated by

$$K_L < -\frac{1}{9}Q^2\mathrm{Cot}[\Psi] - \frac{\mathrm{Tan}[\Psi]}{9Q^2}$$

Extreme Loaded Q Regime

$\rho \approx 1$, the oscillatory threshold occurs before the monotonic; Ψ<0 is excluded, as shown in Fig.2. For $\rho = 1$ & Q>>1, RH_4 is approx.

$$K_L < -\frac{2\mathrm{Csc}[2\Psi]}{Q} + \frac{\mathrm{Tan}[\Psi]}{Q}$$

Intermediate Regime

$\rho \approx \sqrt{Q}$, the operable tuning space is between two competing instabilities, as shown in Fig.3. For $\rho = \sqrt{Q}$ and Q>>1 RH_4 is approximated by

$$K_L < -\frac{1}{2}\sqrt{Q}\mathrm{Csc}[2\Psi] + \frac{1}{4}\sqrt{Q}\mathrm{Tan}[\Psi]$$

Influence of Microphonics

Microphonics (μP) are disturbances of the EM resonance frequency due to mechanical vibrations that couple to the longitudinal MMs. Significantly, at high field, μPs may move the working point Ψ out of the narrow stable region and into one of the unstable regions. This is believed to be the case observed in the ARIEL EACA cryomodule.

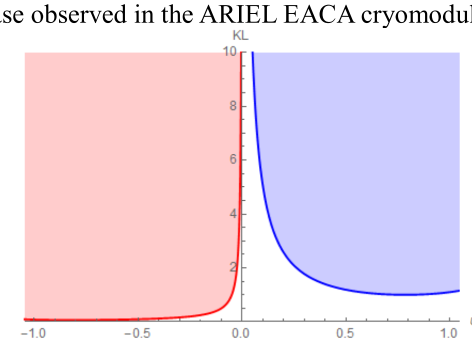

Figure 2: Stability region (white) when ρ≈1.

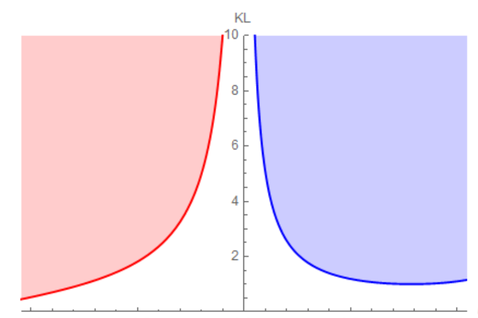

Figure 3: Stability region (white) when $\rho = \sqrt{Q}$.

WITH CONTROL LOOPS

Frequency dependent control loops raise the order of the characteristic polynomial; and the mathematical working quickly becomes stupendous (OED). Therefore, we limit to the case of constant gains, as occurs for low frequency. Another way to reduce the size of the Routh determinants is to retain only the higher powers of Q>>1. For example, RH_4 above changes little when only Q^3 and Q^2 are retained. However, it transpires that the case of constant gains K_a,K_p,K_t>0 leads to a quartic polynomial and is tractable. The working is simplified immensely if we introduce new variables $K_a' = 1+K_a$, $K_p' = 1+K_p$,

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K_t'=1+K_t. Setting the primed variables equal unity recovers the case of no control loops. For brevity we drop the primes (‘). The polynomial coefficients are:

$$a_0 = Q\Omega^2 K_p(K_a K_t - 2K_L \mathrm{Tan}[\Psi] + K_a \mathrm{Tan}[\Psi]^2)$$
$$a_1 = \Omega(2K_a K_p K_t + Q\tau\Omega(K_a + K_p K_t) + 2K_a K_p \mathrm{Tan}[\Psi]^2)$$
$$a_2 = Q\tau^2\Omega^2 + QK_a K_p K_t + 2\tau\Omega(K_a + K_p K_t) + QK_a K_p \mathrm{Tan}[\Psi]^2$$
$$a_3 = \tau(2\tau\Omega + QK_a + QK_p K_t)$$
$$a_4 = Q\tau^2$$

All a_i are automatically positive, except a_0. The monotonic stability condition becomes:

$$K_L < \frac{1}{2}K_a(\mathrm{Cot}[\Psi]K_t + \mathrm{Tan}[\Psi])$$

and is raised significantly by the amplitude and tuning loop gains. Note, K_a and K_t stand in place of 1+K_a and 1+K_t respectively. The general expression for RH$_4$ is:

$$K_L\{\frac{Q\rho K_p(2\rho + Q(K_a + K_p K_t))^2}{K_a + K_p K_t}\} < \{-(Q^2K_a^2K_p^2 + \mathrm{Cot}[\Psi]^4(2\rho K_a + Q(\rho^2 + K_a^2)) (Q\rho^2 + 2\rho K_p K_t + QK_p^2K_t^2) + 2\mathrm{Cot}[\Psi]^2 K_a K_p((2 - Q^2)\rho^2 + Q^2 K_a K_p K_t + Q\rho(K_a + K_p K_t)))\mathrm{Tan}[\Psi]^3\}$$

RH$_4$ is automatically positive for Tan[Ψ]≥0. It will be enlightening to consider special cases.

Classical Regime

We substitute $\rho = Q$. We use the mechanical Q as a surrogate for large feedback gains, and set K_a=K_p=K_t=Q. Retaining only the high powers of Q, leads to the oscillatory threshold condition:

$$K_L < -\frac{2\mathrm{Cot}[\Psi]}{Q} - 2\mathrm{Tan}[\Psi] - \frac{\mathrm{Tan}[\Psi]^3}{Q}$$

Taking a more moderate gain condition $K_a = K_p = K_t = \sqrt{Q}$ yields the threshold:

$$K_L < -2\sqrt{Q}\mathrm{Cot}[\Psi] + \frac{2\mathrm{Tan}[\Psi]}{\sqrt{Q}}$$

Heavily Loaded Regime

We substitute $\rho = 1$. We set K_a=K_p=K_t=Q. Retaining high powers of Q, leads to the oscillatory threshold:

$$K_L < -Q^2\mathrm{Cot}[\Psi] - 2Q\mathrm{Tan}[\Psi] - \mathrm{Tan}[\Psi]^3$$

The gain condition $K_a = K_p = K_t = \sqrt{Q}$ yields:

$$K_L < -Q\mathrm{Cot}[\Psi] - 2\sqrt{Q}\mathrm{Tan}[\Psi] - \mathrm{Tan}[\Psi]^3$$

Intermediate Regime

We substitute $\rho = \sqrt{Q}$. We set K_a=K_p=K_t=Q. Retaining high powers of Q, leads to the oscillatory threshold:

$$K_L < -Q^{3/2}\mathrm{Cot}[\Psi] - \frac{2\mathrm{Tan}[\Psi]}{Q} - \frac{\mathrm{Tan}[\Psi]^3}{\sqrt{Q}}$$

The gain condition $K_a = K_p = K_t = \sqrt{Q}$ yields:

$$K_L < -2\sqrt{Q}\mathrm{Cot}[\Psi] - \frac{2\mathrm{Tan}[\Psi]}{Q} - \frac{\mathrm{Tan}[\Psi]^3}{\sqrt{Q}}$$

NO TUNING LOOP

The results in the previous section, have the demerit that the contributions of the individual loops is not apparent. We now state results for “no tuning loop” or the tuner very much slower than the cavity (loaded) time constant. We set K_t=0. The residual is the contribution of phase and amplitude loops, which we take to have equal gains.

The monotonic instability threshold occurs for positive detuning, Tan[Ψ]>0.

$$K_L < \mathrm{Csc}[2\Psi]K_a$$

The Routh determinant RH$_4$ may become negative for negative detuning Tan[Ψ]<0.

Classical Regime

We substitute $\rho = Q$. We use the mechanical Q as a surrogate for large feedback gains, and set K_a=K_p=Q. Retaining only the high powers of Q, leads to the oscillatory threshold condition:

$$K_L < -\mathrm{Cot}[\Psi] - \frac{\mathrm{Tan}[\Psi]}{2Q} - \frac{\mathrm{Tan}[\Psi]^3}{4}$$

Taking a more moderate gain condition $K_a = K_p = \sqrt{Q}$ yields the threshold:

$$K_L < -\frac{1}{4}Q\mathrm{Cot}[\Psi] + \frac{\mathrm{Tan}[\Psi]}{2}$$

Heavily Loaded Regime

We substitute $\rho = 1$. We set K_a=K_t=Q. Retaining high powers of Q, leads to the oscillatory threshold:

$$K_L < -\frac{1}{4}Q\mathrm{Cot}[\Psi] - \frac{1}{2}Q\mathrm{Tan}[\Psi]$$

The gain condition $K_a = K_p = \sqrt{Q}$ yields:

$$K_L < -\frac{1}{2}\mathrm{Csc}[2\Psi] - \frac{\mathrm{Tan}[\Psi]}{4} + \frac{\mathrm{Tan}[\Psi]}{Q}$$

Intermediate Regime

We substitute $\rho = \sqrt{Q}$. We set K_a=K_p=Q. Retaining high powers of Q, leads to the oscillatory threshold:

$$< -\frac{1}{4}\sqrt{Q}\mathrm{Cot}[\Psi] - \frac{1}{2}\sqrt{Q}\mathrm{Tan}[\Psi] - \frac{1}{4}\sqrt{Q}\mathrm{Tan}[\Psi]^3$$

The gain condition $K_a = K_p = \sqrt{Q}$ yields:

$$K_L < -\frac{\mathrm{Cot}[\Psi]}{\sqrt{Q}} - \frac{\mathrm{Tan}[\Psi]}{2Q^{3/2}} - \frac{\mathrm{Tan}[\Psi]^3}{4\sqrt{Q}}$$

CONCLUSION

This short paper exemplifies use of the Routh Hurwitz analysis by application to the ponderomotive instability of a generator driven SRF cavity with (or without) control loops. The roots of the characteristic provide both the thresholds and the growth rates. Notably, we draw attention to ρ=τΩ, the product of cavity (loaded) time constant and mechanical mode frequency as a relevant parameter by which to map out the domains of stability. The range of applicability is extended compared with Schulze. Importantly, we discover that the oscillatory instability can occur before the monotonic in the heavily loaded regime.

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