SELF-CONSISTENT MODEL OF 3-DIMENSIONAL TIME-DEPENDENT ELLIPSOID

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Abstract

A self-consistent treatment of a 3-dimensional time-dependent ellipsoid with negligible emittance is performed. Envelope equations describing the evolution of an ellipsoid boundary are obtained. For a complete model it is required that the initial particle momenta be a linear function of the coordinates. Numerical examples and verification of the problem by a 3-dimensional particle-in-cell simulations are given.

1 INTRODUCTION

A bunched beam in an accelerating field is often approximated by a uniformly charged ellipsoid. However, self-consistent solutions corresponding to such an ellipsoid are valid only in special cases. Time - independent solutions for an azimuthally-symmetric ellipsoid (spheroid) were treated in Refs. [1], [2] and time -dependent solutions for the same ellipsoid were found in Ref. [3]. It is well known that there is no 3-dimensional self-consistent solution for a time-dependent uniformly charged ellipsoid, similar to KV distribution [4]. In this paper we consider the existence of a solution for a 3D time-dependent ellipsoid with zero phase space volume.

2 TIME-DEPENDENT ELLIPSOID IN SELF-CONSISTENT FIELD

Consider the evolution of an initially uniformly charged ellipsoid in the rest system of coordinates with applied focusing potential

$$U_{ext}(x, y, z, t) = G_x(t)\frac{x^2}{2} + G_y(t)\frac{y^2}{2} + G_z(t)\frac{z^2}{2}, \quad (1)$$

where $G_x(t)$, $G_y(t)$, $G_z(t)$ are time-dependent gradients of the focusing field in 3 directions. External focusing fields are linear functions of coordinates:

$$E_x^{(ext)} = -G_x(t)\,x, \quad E_y^{(ext)} = -G_y(t)\,y, \quad E_z^{(ext)} = -G_z(t)\,z. \quad (2)$$

The potential of the uniformly charged ellipsoid in free space is given by [5]

$$U_b = -\frac{\rho R_x R_y R_z}{4\,\varepsilon_o}\int_o^\infty \frac{\left(\frac{x^2}{R_x^2+s}+\frac{y^2}{R_y^2+s}+\frac{z^2}{R_z^2+s}\right)ds}{\sqrt{(R_x^2+s)(R_y^2+s)(R_z^2+s)}}, \quad (3)$$

$$\rho = \frac{3}{4\pi}\frac{Q_e}{R_x R_y R_z}, \quad (4)$$

where Q_e is the charge, R_x, R_y, R_z are semi-axes and ρ is the space charge density of the ellipsoid. The components of the electrostatic field of the ellipsoid are linear functions of the coordinates:

$$E_\xi^{(b)} = -\frac{\partial U_b}{\partial \xi} = \frac{\rho\, M_\xi}{\varepsilon_o}\,\xi, \quad (5)$$

$$M_\xi = \frac{1}{2}\int_o^\infty \frac{R_x R_y R_z\, ds}{(R_\xi^2+s)\sqrt{(R_x^2+s)(R_y^2+s)(R_z^2+s)}}, \quad (6)$$

where $\xi = x, y, z$.

Consider the dynamics of an arbitrary element inside the ellipsoid with coordinates (x, x+ dx), (y, y + dy), (z, z + dz) which contains dN (x, y, z) particles. Assume that the ellipsoid remains uniformly populated, therefore the equations of particle motion under the external field and space charge forces of the ellipsoid are linear:

$$\begin{cases} \frac{dx}{dt} = \frac{p_x}{m} \\ \frac{dp_x}{dt} = -qG_x(t)x + q\frac{\rho(t)\,M_x(t)}{\varepsilon_o}x \end{cases}, \quad (7)$$

similarly for the y and z directions. The general solution $x(t)$, $p_x(t)$ of the set of linear differential equations of the first order, Eqs. (7), are linear combinations of the initial conditions x_o, p_{xo}:

$$\begin{vmatrix} x(t) \\ p_x(t) \end{vmatrix} = \begin{vmatrix} a_{11}(t) & a_{12}(t) \\ a_{11}(t) & a_{12}(t) \end{vmatrix} \begin{vmatrix} x_o \\ p_{xo} \end{vmatrix}, \quad (8)$$

where $a_{ij}(t)$, i,j =1, 2 are coefficients of the solution matrix.. Similar solutions are valid for the y and z directions. Let us introduce an additional requirement that the initial particle momenta are linear functions of the coordinates:

$$p_{xo} = \alpha_x \cdot x_o, \quad p_{yo} = \alpha_y \cdot y_o, \quad p_{zo} = \alpha_z \cdot z_o. \quad (9)$$

In this case the solution, x(t), is a linear function of the initial particle position:

$$x(t) = a_{11}(t)\, x_o + a_{12}(t)\, \alpha_x x_o = c_x(t) \cdot x_o, \quad (10)$$

and similarly, $y(t) = c_y(t)\cdot y_o$, $z(t) = c_z(t)\cdot z_o$. At a fixed moment of time, t, the volume of a selected element, dV(t) = dx(t) dy(t) dz(t), is connected with the initial volume, $dV_o = dx_o\, dy_o\, dz_o$, by the linear relationship $dx(t)\, dy(t)\, dz(t) = c_x(t)\, c_y(t)\, c_z(t)\, dx_o\, dy_o\, dz_o$, or

$$dV(t) = c(t)\, dV_o. \quad (11)$$

The number of particles inside the selected element is conserved, dN = const, because no one particle can penetrate the boundary of an element because of the linear transformation of particle positions, Eq. (10). Therefore, the particle density, $\rho(t) = dN/dV(t)$, is connected with the initial density, $\rho_o = dN/dV_o$, by the linear equation $\rho(t) = \rho_o\, dV_o/dV(t)$, or

$$\rho(x,y,z,t) = \frac{\rho(x_o,y_o,z_o,0)}{c(t)} \quad . \qquad (12)$$

Eq. (12) indicates that the initially uniformly populated ellipsoid, $\rho(x_o,y_o,z_o,0) = \text{const}$, remains uniformly populated while propagating in linear field. Space charge density of the ellipsoid, $\rho(x,y,z,t)$, depends only on time according to Eq. (12) and is not a function of coordinates x, y, z. Such an ellipsoid delivers linear space charge forces according to Eqs. (5), (6). Therefore, the original suggestion about particle motion in a linear field is proved to be correct.

Due to the absence of momentum spread in the beam, particles at the surface of the ellipsoid remain there during the evolution of the ellipsoid, and envelope equations can be written as equations for maximum extended particles with coordinates $x = R_x$, $y = R_y$, $z = R_z$:

$$\frac{d^2R_x}{dt^2} + \frac{q\,G_x(t)}{m} R_x - \frac{3}{4\pi}\frac{q}{m}\frac{Q_e}{\varepsilon_o}\frac{M_x(R_x, R_y, R_z)}{R_y\,R_z} = 0, \qquad (13)$$

$$\frac{d^2R_y}{dt^2} + \frac{q\,G_y(t)}{m} R_y - \frac{3}{4\pi}\frac{q}{m}\frac{Q_e}{\varepsilon_o}\frac{M_y(R_x, R_y, R_z)}{R_x\,R_z} = 0, \qquad (14)$$

$$\frac{d^2R_z}{dt^2} + \frac{q\,G_z(t)}{m} R_z - \frac{3}{4\pi}\frac{q}{m}\frac{Q_e}{\varepsilon_o}\frac{M_z(R_x, R_y, R_z)}{R_x\,R_y} = 0\,. \qquad (15)$$

3 DRIFT OF ELLIPSOID IN FREE SPACE

The expansion of the ellipsoid in a drift space is described by Eqs. (13) - (15) with $G_x = G_y = G_z = 0$. In Figs. 1, 2 numerical results of the drift of an ellipsoid with Q_e = 3 nK with the initial semi-axes values $R_x = 2$ cm, $R_y = 1$ cm, $R_z = 4$ cm and longitudinal velocity of $\bar{\beta}_z = 0.01$ are presented. Numerical calculations were performed using the 3D particle-in-cell code BEAMPATH [6] utilizing $2\cdot10^4$ particles on the grid 1/2 N_x x N_y x N_z = 64 x 128 x 512. The difference in analytical and numerical values of the ellipsoid sizes are within 3% of each other.

4 APPLICATION TO PARTICLE DYNAMICS IN A LINAC

The particle motion in an RF field with uniform focusing is described by the Hamiltonian [7]:

$$H = \frac{p_x^2 + p_y^2}{2\,m\,\gamma} + \frac{p_z^2}{2\,m\,\gamma^3} + q\,U_{ext} + q\,\frac{U_b}{\gamma^2}, \qquad (16)$$

$$U_{ext} = \frac{E}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z z) - \sin\varphi_s + k_z z\cos\varphi_s] + G\frac{r^2}{2}, \qquad (17)$$

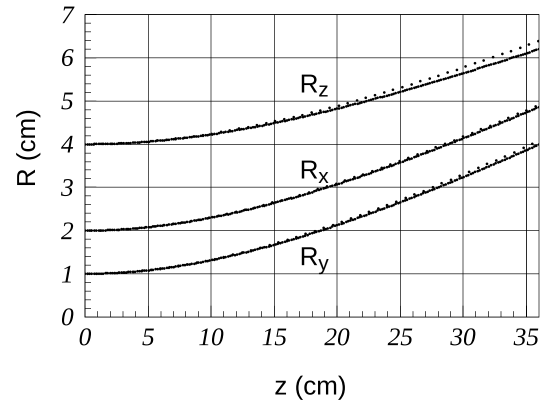

Fig. 1. Envelopes of a uniformly populated ellipsoid in a drift space: solid lines - PIC simulation, dotted lines - analytical solution of Eqs. (13) - (15).

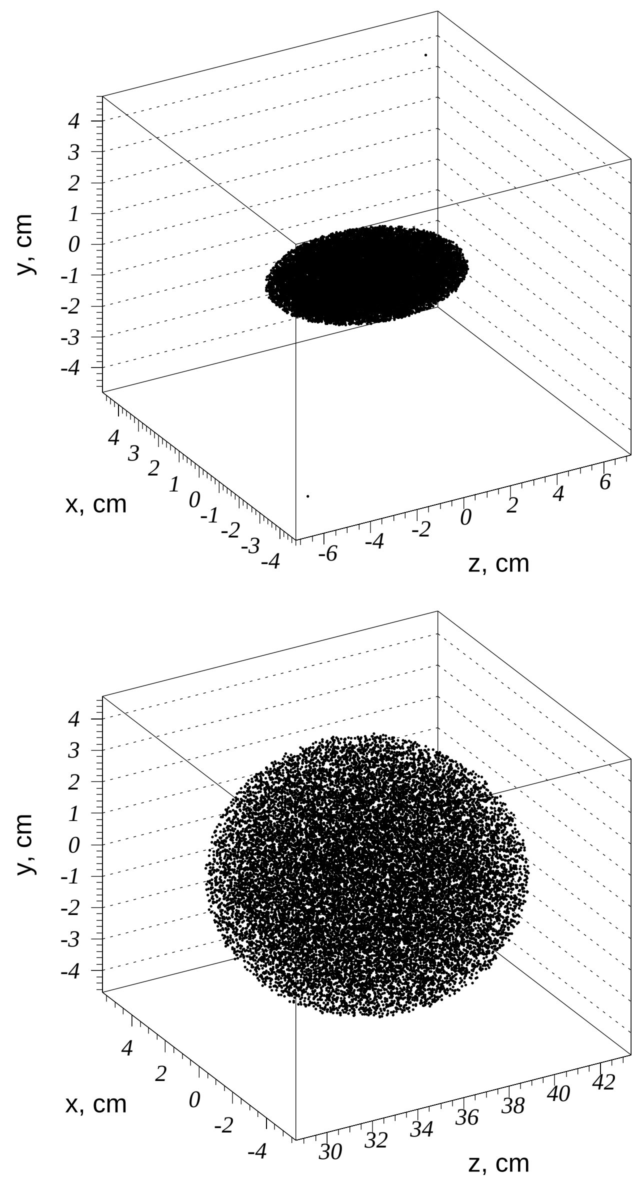

Fig. 2. Uniformly populated ellipsoid in drift space: (a) t = 0, (b) t = $1.2\cdot10^{-7}$ sec.

where p_x and p_y are transverse particle momenta, $p_z = p - p_s$ is the deviation from the longitudinal momentum, z is the deviation from the position of a synchronous particle, E is the amplitude of the accelerating field, φ_s is the synchronous phase, $k_z = 2\pi/(\beta_s\lambda)$ is the wave number, λ is the wavelength, G is the constant gradient of the focusing field, and r is the particle radius. The potential of the external field, U_{ext}, is a nonlinear function of the coordinates z, r. In the vicinity of a synchronous particle, $\omega z/v_s << 1$, the following expansion is valid:

$$\sin(\varphi_s - \frac{\omega}{v_s} z) \approx \sin\varphi_s - (\frac{\omega}{v_s} z)\cos\varphi_s - \frac{1}{2}(\frac{\omega}{v_s} z)^2 \sin\varphi_s . \quad (18)$$

The approximation, Eq. (18), is valid for longitudinal particle oscillations, much smaller than the separatrix size. In addition, consider the radial deviation to be much smaller than the bunch period $r << \beta_s\lambda$ then we can assume

$$I_o\left(\frac{\omega r}{\gamma v_s}\right) \approx 1 + \frac{1}{4}\left(\frac{\omega r}{\gamma v_s}\right)^2 . \quad (19)$$

Under these restrictions, the potential, Eq. (17), becomes:

$$U_{ext} = G_z \frac{z^2}{2} + G \frac{r^2}{2}\left[1 - \frac{G_z}{2\gamma^2 G}\frac{\sin(\varphi_s - k_z z)}{\sin\varphi_s}\right] , \quad (20)$$

$$G_z = \frac{\omega E |\sin\varphi_s|}{v_s} . \quad (21)$$

Potential, Eq. (20), depends on phase of particle in RF field. For small accelerating gradient, $G_z/(2\gamma^2 G) << 1$, the potential, Eq. (20), is close to that of Eq. (1) and the envelope equations (13) - (15) describe the evolution of a small ellipsoidal bunch in a constant external field. Special solutions $R_x'' = R_y'' = R_z'' = 0$ give the conditions for a stationary (time-independent) bunch, which is in equilibrium with the external field:

$$G_\xi = \frac{3}{4\pi}\frac{Q_e}{\varepsilon_o}\frac{M_\xi(R_x, R_y, R_z)}{R_x R_y R_z} , \quad \xi = x, y, z . \quad (22)$$

In Fig. 3 the results of beam dynamics with Q = 1.4 nK in a channel with G = 3.6 kV/cm^2, G_z= 0.58 kV/cm^2, λ = 8.57 m, β = 0.0178 are presented. The values of $R_x = R_y$ = 0.5 cm, R_z = 2 cm correspond to a stationary bunch. The initial conditions for an ellipsoidal bunch were selected to be R_x = 0.4 cm, R_y = 0.6 cm, R_z = 1.8 cm. Deviation from the stationary solution results in oscillations around equilibrium, while the ellipsoid remains uniformly populated.

5 REFERENCES

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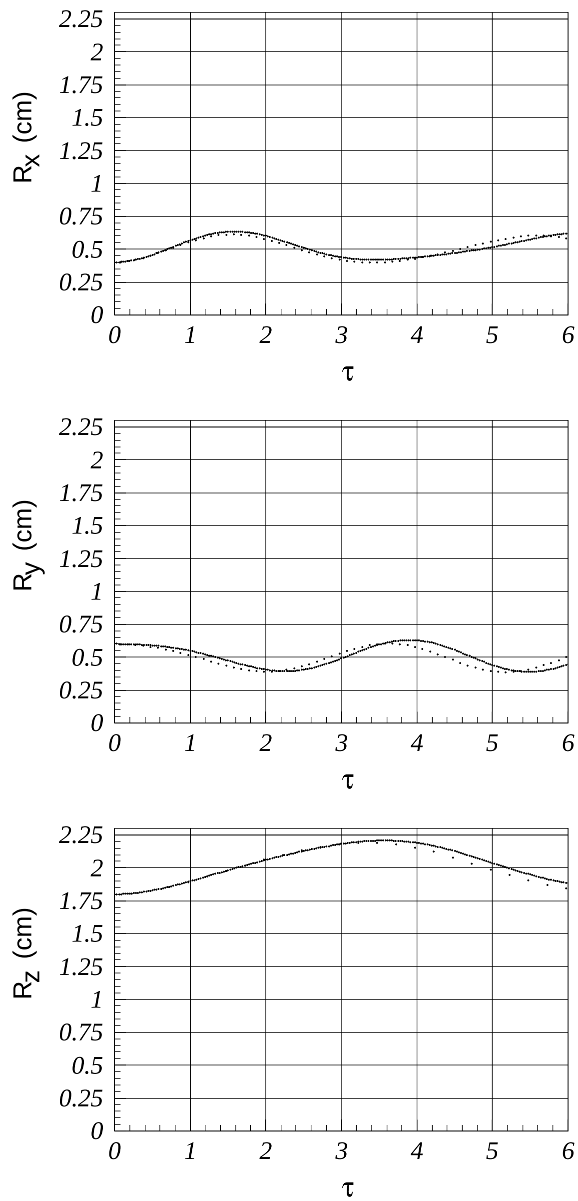

Fig. 3. Envelopes of an ellipsoid in an accelerating-focusing channel, $\tau = tc/\lambda$; solid lines - PIC simulation, dotted lines - analytical solution.