DESIGN EQUATIONS IN AN ALVAREZ-TYPE PROTON LINEAR ACCELERATOR

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Abstract

The accurate beam dynamics equations for an Alvarez-type linac can be used to give cell-by-cell design data. In particular, the cell lengths can be chosen such that a reference particle follows any phase law, in a tank where the mean E-field can be chosen arbitrarily, and where drift tube diameters and apertures may also vary. The "S term" for a particle in a gap is derived, and correction to both S and T in the design for a non-zero inner profile radius of curvature is discussed. Finally, the axial equations of motion in such a linac are stated.

Introduction

Accurate beam dynamics equations[1,2] for an Alvarez-type linac show how the motion of a particle depends on small phase changes experienced during acceleration in a gap. This dependency can be utilized to give the longitudinal dimensions of a linac such that a reference particle on the axis obeys any required phase law, even where the mean E-field has been chosen from separate requirements. Thus the phase law could be chosen from general considerations of phase motion (for example, the phase at the gap centre of cell n, $\varphi_c(n)$, may be such that the bucket area or the energy spread along the linac is constant), and it is conceivable that the amelioration of space charge effects could be included in such a law. There are several possibilities in the choice of mean E-field: for example, a constant mean E-field, for simplicity and ease of setting-up; or a field $\bar{E}(n)$ to give a limiting surface E-field at the drift tubes of each cell, to give the maximum utilization of the linac; or again to choose $\bar{E}(n)$ to reduce the effects of space charge on the beam longitudinal motion. In choosing either of the above quantities, the concept of acceleration rate is useful only in so far that it gives a guide to the order of magnitude of $\bar{E}$, or φ_c. For example, acceleration rate can be used to make a first guess at cell lengths, and the number and location of the centre cell of a tank, from which $\bar{E}$, or φ_c could be chosen from $\bar{E}(n)T(n)\cos\varphi_c(n) = k(n)$. With $\bar{E}(n)$ and $\varphi_c(n)$ chosen, the dimensions, and resulting motion of the beam, are found on a cell-by-cell basis.

Cell-by-Cell Design Equations

The cell-by-cell computations of a linac tank depend on whether the tank is regarded as N whole cells (the simpler case), or more generally 2N half cells. In each case the resonance condition is expressed by the dimensionless equations $g/L = f[L/\lambda;\ (\text{other dimensions})/\lambda]$ or $g/2/L/2 = f[L/\lambda;\ (\text{other dimensions})/\lambda]$. The first equality suffices when there is geometric symmetry in a gap, so the electrical and geometric centres coincide. The second recognizes that the cell halves may be different because of change of drift tube diameter, or profile or aperture, and hence the geometric gap centre and the electrical centre are not the same. (There will, of course, be as many g/2/L/2 laws as changes in "other dimensions".) In the first method whole values of L and g are found, and it will be seen that the method requires the ability to move the gap (by very small amounts) along the length of the cell, and that doing so has negligible effect on frequency; the second method finds "half gaps" and assumes that joining them together at the electrical centres again has negligible effect on frequency. In the following, a working definition of electrical centre is the plane where the voltages over the two "half cells" are equal (rather than the more rigorous, but less usable definition, $\int E_z \sin k_z z\, dz = 0$).

Consider the cell n of a linac tank, Fig. 1, in which the phase at the electrical centre (which is generally not the geometric centre) is $\varphi_{c,n}$. The phases at input and output to this cell are

$$-\pi + \tfrac{1}{2}(\varphi_{c,n} + \varphi_{c,n-1}) \text{ and } +\pi + \tfrac{1}{2}(\varphi_{c,n+1} + \varphi_{c,n}),$$

respectively, and the phase change across the cell is $2\pi + \tfrac{1}{2}(\varphi_{c,n+1} - \varphi_{c,n-1})$.

Let us take, as the first method, a tank where diameters are equal, so that the gap is symmetrical: then $g_{n,1}/2 = g_{n,2}/2 = g/2$ (and $\varphi_c = \varphi_{\text{geom centre}}$). It can be shown from reference 2) that the phase changes across the cell are as follows:

$$\varphi_{c,n} - \varphi_{in} = \frac{2\pi}{\beta_{in}\lambda}(SL_{n,1}) + \frac{\Delta\varphi}{2} - \frac{x}{2} = \pi + \frac{1}{2}(\varphi_{c,n} - \varphi_{c,n-1}) \quad (1)$$

$$\varphi_{out} - \varphi_{c,n} = \frac{2\pi}{\beta_{out}\lambda}(SL_{n,2}) + \frac{\Delta\varphi}{2} + \frac{x}{2} = \pi + \frac{1}{2}(\varphi_{c,n+1} - \varphi_{c,n}) \quad (2)$$

Since $L_n = SL_{n,1} + SL_{n,2}$, adding equations (1)

and (2) gives

$$L_n = \bar{\beta}\lambda - \frac{\lambda}{2\pi}\left\{\bar{\beta}\Delta\varphi + \frac{x}{2}\Delta\beta\right\} + \frac{\lambda}{4\pi}\left\{\beta_{in}\left(\varphi_{c,n+1} - \varphi_{c,n-1}\right) + \Delta\beta\left(\varphi_{c,n+1} - \varphi_{c,n}\right)\right\} \quad (3)$$

where $\bar{\beta}$ is the mean value, and $\Delta\beta$ the change in β across the cell, and

$$\Delta\varphi = -V_o \frac{(1-\beta_c^2)^{3/2}}{W_o\beta_c}\frac{dT}{d\beta_c}\sin\varphi_c \; ; \quad x = -V_o \frac{(1-\beta_c^2)^{3/2}}{W_o\beta_c}\frac{dS}{d\beta_c}\cos\varphi_c \quad (4)$$

where β_c is the β-value at the gap centre; in the absence of more accurate empirical expressions which could otherwise be used, T and S and their derivatives are given by, as for a uniform field at r = a,

$$T(r=0) = \frac{\sin\vartheta}{\vartheta}\frac{1}{I_o(k_r a)}, \quad S(r=0) = \frac{1}{\vartheta}\left(1 - \frac{\cos\vartheta}{I_o(k_r a)}\right); \quad \vartheta = \frac{\pi g}{\beta_c\lambda} \quad (5)$$

(the derivation of S, and the modification to T and S for non-zero profile inner radius, are left to Sections 3 and 4). Since $V_o = \bar{E}_n L_n$, equations (3) and (4) lead to

$$L_n = \frac{\bar{\beta}\lambda + \frac{\lambda}{4\pi}\left\{\beta_{in}\left(\varphi_{c,n+1} - \varphi_{c,n-1}\right) + \Delta\beta\left(\varphi_{c,n+1} - \varphi_{c,n}\right)\right\}}{\left\{1 - \frac{\lambda}{2\pi}\bar{E}_n\frac{(1-\beta_c^2)^{3/2}}{W_o\beta_c}\left[\bar{\beta}\frac{dT}{d\beta_c}\sin\varphi_{c,n} + \frac{\Delta\beta}{2}\frac{dS}{d\beta_c}\cos\varphi_{c,n}\right]\right\}} \quad (6)$$

Equation (6) shows clearly the dependence of L_n on the separate choice of $\varphi_{c,n}$ and $\bar{E}$. With the help of the equations for energy gain to gap centre and across the whole gap, namely:

$$\Delta W_1 = \frac{\Delta W}{2}\left(1 + \frac{S}{T}\tan\varphi_{c,n}\right); \quad \Delta W = \bar{E}_n L_n T(\beta_c)\cos\varphi_{c,n} \quad (7)$$

to give β_{out}, $\bar{\beta}$, $\Delta\beta$ and β_c, equation (6) can be solved iteratively to give L_n, and g_n. Note that since $SL_{n,1} \neq SL_{n,2}$, the gap is not symmetrically placed along the cell: a fact which we assume does not affect frequency.

In the second case $g_{n,1}/2 \neq g_{n,2}/2$. Nevertheless, it remains convenient to use the uniform gap field equations for T and S. The voltage in each electrical half of the cell n is the same. The cell may then be regarded as the superposition of two cells (Fig. 2): the first half of a cell ($2SL_{n,1}$, $2g_{n,1}/2$) and the second half of a cell ($2SL_{n,2}$, $2g_{n,2}/2$). The voltage V_o in each cell (= 2 x voltage in the actual half gap) = $\bar{E}_n(SL_{n,1} + SL_{n,2})$, and for which a first guess can be made as indicated in Section 1. In the first half of the cell $SL_{n,1}$ (and hence $g_{n,1}/2$) is given by

$$SL_{n,1} = \frac{\beta_{in}\lambda}{2\pi}\left\{\pi + \frac{1}{2}\left(\varphi_{c,n} - \varphi_{c,n-1}\right) - \frac{\Delta\varphi_1}{2} + \frac{x_1}{2}\right\} \quad (8)$$

where $\Delta\varphi_1$, x_1 are given as in equations (4) and (5), but with $\vartheta = \pi g_{n,1}/\beta_c\lambda$, $a = a_1$. The value β_c at the centre (= electrical centre of the complete gap) is given, similarly to equation (7), by

$$\Delta W_1 = W_o\left\{\left(1-\beta_c^2\right)^{-1/2} - \left(1-\beta_{in}^2\right)^{-1/2}\right\} = \frac{V_o}{2}\left[T\left(g_{n,1}\right)\cos\varphi_{c,n} + S\left(g_{n,1}\right)\sin\varphi_{c,n}\right]_{\beta=\beta_c}. \quad (9)$$

In the second half of the second cell, $SL_{n,2}$ (and hence $g_{n,2}/2$) is given by

$$SL_{n,2} = \frac{\beta_{out}\lambda}{2\pi}\left\{\pi + \frac{1}{2}\left(\varphi_{c,n+1} - \varphi_{c,n}\right) - \frac{\Delta\varphi_2}{2} - \frac{x_2}{2}\right\} \quad (10)$$

where $\Delta\varphi_2$, x_2 are given as in equations (4) and (5), but with $\vartheta = \pi g_{n,2}/\beta_c\lambda$, $a = a_2$. The value of β_c is already as given from equation (9), and β_{out} is given by the energy gain for the second half of a gap:

$$\Delta W_2 = W_o \left\{ \left(1 - \beta^2_{out}\right)^{-1/2} - \left(1 - \beta^2_c\right)^{-1/2} \right\} = \tag{11}$$

$$= \frac{V_o}{2} \left[T\left(g_{n,2}\right) \cos \varphi_{c,n} - S\left(g_{n,2}\right) \sin \varphi_{c,n} \right]_{\beta = \beta_c} .$$

The length of the cell is again $L_n = SL_{n,1} + SL_{n,2}$, and $g_n = g_{n,1}/2 + g_{n,2}/2$. With each estimate of V_o ($= \bar{E}L_n$), the above equations can be solved for a new value of L_n until an acceptably small difference is obtained [taken as $L_n(I) - L_n(I-1) \leq 0.002$ cm]. In practice the process converges rapidly, and only about three iterations are required. Other dimensions follow immediately, for example drift tube lengths ($= SL_{n,2} + SL_{n+1,1} - g_{n,2}/2 - g_{n+1,2}/2$), and distance between gap (electrical) centres. $SL_n = SL_{n,2} + SL_{n+1,1}$. Also available are the distances between geometric and electric centres: for the gap $= \frac{1}{4}|g_{n,1} - g_{n,2}|$, for the drift tube $= \frac{1}{2}(SL_{n+1,1} - SL_{n,2}) + \frac{1}{4}(g_{n,2} - g_{n+1,1})$: these are typically 0.01 to 0.02 cm and 0.01 cm, respectively, over the range 0.7 to 20 MeV. Note that since $g_{n,1}/2 \neq g_{n,2}/2$, the equivalent uniform E-fields are different and are in the ratio $g_{n,1}/g_{n,2}$: this is typically of the order 1.02 - 1.03. The effect on frequency of adding two half-cells at the electrical centre with such a field imbalance is unknown, but it is clearly very much less than the 2 - 3% field difference. (It is worth remarking that in the past, compounding a linac tank from data on half-cell model measurements or mesh computations has always been done with no reported frequency perturbation.)

The S Coupling Term

The S term is defined by

$$S(\beta_c, r) = 2 \int_0^\infty E_z(z,r,t) \sin k z \, dz \Big/ \int_{-\infty}^{+\infty} E_z(z,r,t) \, dz \tag{12}$$

where $k = 2\pi/\beta_c\lambda$ and the uniform field value reported in reference 1) is incorrect. It is therefore of interest to derive S, as follows. Consider a uniform gap g, which is limited by two circular semi-infinite cylindrical aperture in the z direction. It can be shown[3]) that the fields for $r < a$ (where 2a is the aperture) are given approximately for not too high frequencies, by

$$E(r,z) = E_o \left[1 - 2 \sum_1^\infty \frac{J_o\left(\frac{r}{a} j_n\right) \cos h\left(\frac{j'_n}{a} z\right)}{j_n J_1(j_n)} \right. \tag{13}$$

$$\left. \times \exp\left(-\frac{g}{2a} j'_n\right) \right] \quad |z| < \frac{g}{2}$$

$$= E_o \left[2 \sum_1^\infty \frac{J_o\left(\frac{r}{a} j_n\right) \sin h\left(\frac{g}{2a} j'_n\right)}{j_n J_1(j_n)} \right.$$

$$\left. \times \exp\left(-\frac{|z|}{a} j'_n\right) \right] \quad |z| > \frac{g}{2} .$$

Where E_o is the field at $r = a$, assumed uniform, and j_n is the n^{th} root of $J_o(x) = 0$, and $j'^2_n = j^2_n - (\omega^2 a^2/c^2)$. Substituting in equation (12), and performing the integrations

$$\int_0^{g/2} + \int_{g/2}^\infty$$

leads to

$$S(\beta_c, r) = \frac{2}{g} \left\{ \frac{1}{k} \left(1 - \cos \frac{kg}{2}\right) \right.$$

$$+ 2 \sum_1^\infty \frac{k \cos\left(\frac{kg}{2}\right) J_o\left(\frac{r}{a} j_n\right)}{j_n J_1(j_n) \left[\left(\frac{j'_n}{a}\right)^2 + k^2\right]}$$

$$\left. - 2 \sum_1^\infty \frac{k \, J_o\left(\frac{r}{a} j_n\right) \exp\left(-\frac{g}{2a} j'_n\right)}{j_n J_1(j_n) \left[\left(\frac{j'_n}{a}\right)^2 + k^2\right]} \right\}$$

But by performing similar integrations for

$$T(r) = \frac{\sin\left(\frac{kg}{2}\right) I_o\left(k_r r\right)}{\left(\frac{kg}{2}\right) I_o\left(k_r a\right)}$$

it can be seen that the second term in the

bracket =

$$\cos\left(\frac{kg}{2}\right) \frac{I_o(k_r a) - I_o(k_r r)}{k\, I_o(k_r a)} .$$

Hence, for not too high β

$$S(\beta_c, r) = \frac{1}{\left(\frac{kg}{2}\right)}\left[1 - \frac{I_o(k_r r)}{I_o(k_r a)} \cos\left(\frac{kg}{2}\right)\right] \qquad (14)$$

$$- 2 \sum_{1}^{\infty} \frac{\left(ka\right)^2 J_o\left(\frac{r}{a} j_n\right) \exp\left(-\frac{g}{2a} j_n'\right)}{\left(\frac{kg}{2}\right) j_n J_1\left(j_n\right)\left[j_n'^2 + \left(ka\right)^2\right]}$$

in which the second term is rapidly converging. The second term is largest for r = 0, and decreases with increasing energy. Computations on the new Tank 1 design of the Rutherford Lab. P.L.A. show that at greatest (at 0.515 MeV) the first term of the sum is < 1% of the first term of equation (14) and the remainder is of the order 10^{-5} of it. The sum can be ignored so that S(r) is given only by the first term of equation (14).

Corrections to T and S for a Finite Inner Profile Radius of Curvature

Values for T and S given by equations (5) are for a uniform field in the gap at r = a which corresponds to a profile as shown typically in Fig. 3 obtained from an analytical extension of equation (13) for r > a in the gap region; there the profile radius of curvature is zero at the hole entrance. The effect of a finite radius of curvature is, of course, to increase the beam interaction region, but in a way which is difficult to describe analytically. However, mesh calculations have been made on the Rutherford Laboratory new Tank 1 design (by A Katz of C.E.N., Saclay, to whom the authors express their thanks) which cover the range 0.515 to 10 MeV in 42 cells, g/L = 0.234 - 0.337, a = 0.677 cm - 1.25 (cell 25) - 1.25 (cell 42), and these have been compared with the simple formulae for T and S. The best approximations to the two simple formulae were found to be

$$\text{at } r = 0 \begin{cases} T_{actual} = T(g+R) \\ S_{actual} = S(g) \end{cases}, \qquad (15)$$

R is the inner profile radius

and were better than 1%(T) and 2%(S). (Good accuracy is more important in T, since S itself is a correction.) Off axis, T(g+R, r) agreed closely with computed values, to < 1%, whilst S(g,r) remained within 4% of the computed values, the accuracy being worst as r → a. The accuracy is not so good for $dT/d\beta$ [typically 5% smaller than computed from Eq. (5)], and even worse for $dS/d\beta$ [typically up to 10% smaller than computed from Eq. (5)]. But since the corresponding terms are only corrections, the accuracy is acceptable. The parameters of the Tank 1 design are sufficiently varied for the formulae (15) to be considered generally applicable. Thus in the equations (5) and in all the design equations of Section 2, all terms in T have (g+R), and in S have g alone.

Axial Equations of Motion

The equations of Section 2 also give the distance between electrical centres, $SL_n = SL_{n,2} + SL_{n+1,1}$.

In method 1:

$$SL_n = \frac{\beta_{o\,n}\lambda}{2\pi}\left[2\pi - \frac{1}{2}\left(\Delta\varphi_n + \Delta\varphi_{n+1}\right)\right. \qquad (16)$$

$$\left. + \frac{1}{2}\left(x_{n+1} - x_n\right)\right] + \frac{\beta_{o\,n}\lambda}{2\pi}\left(\varphi_{c,n+1} - \varphi_{c,n}\right)$$

where $\beta_{o,n}$ is the output β of the reference particle at cell n. For any particle, then, the exact equation of phase w.r.t. r.f. is

$$\varphi_{c,n+1} = \varphi_{c,n} + 2\pi\left(\frac{SL_n}{\beta_o\lambda} - 1\right) \qquad (17)$$

$$+ \frac{1}{2}\left(\Delta\varphi_{n+1} + \Delta\varphi_n\right) - \frac{1}{2}\left(x_{n+1} - x_n\right)$$

where β_o is the output β of the general particle at cell n, and $\Delta\varphi, x$ refer to that particle. It is interesting to note that the phase at gap centre n + 1 is not known until its motion is already known by $\Delta\varphi_{n+1}$, x_{n+1}. But $\Delta\varphi, x$ are slowly varying, so that equation (17) can be approximated to

$$\varphi_{c,n+1} \approx \varphi_{c,n} + 2\pi\left(\frac{SL_n}{\beta_o\lambda} - 1\right) + \Delta\varphi_n \qquad (18)$$

which is the equation given in reference 1). Equation (18) together with equations (7) give the complete motion.

In method 2 the phase equation is

$$\varphi_{c,n+1} = \varphi_{c,n} + 2\pi\left(\frac{SL_n}{\beta_o\lambda} - 1\right) \qquad (19)$$

$$+ \frac{1}{2}\left(\Delta\varphi_{n,2} + \Delta\varphi_{n+1,1}\right) - \frac{1}{2}\left(x_{n+1,1} - x_{n,2}\right)$$

where the quantities $\Delta\varphi_{n,r}$, $x_{n,r}$ refer to whole gaps $2(g_{n,r}/2)$. With the half-gap energy equation of Section 2, the change of energy for the whole

gap is now

$$\Delta W = \frac{V_0}{2}\left\{\left[T(g_{n,1};\beta_c) + T(g_{n,2};\beta_c)\right]\cos\varphi_{c,n} + \left[S(g_{n,1};\beta_c) - S(g_{n,2};\beta_c)\right]\sin\varphi_{c,n}\right\} \quad (20)$$

where T and S refer to whole gaps $g_{n,1}$ or $g_{n,2}$. Since $g - g_{n,1}$ or $g - g_{n,2}$ are small (typically in the range 0.03 to 0.04 cm, whereas g is typically at least 1.5 cm in a 200 Mc/sec structure), equation (20) can be expanded to the first order to give

$$\Delta W = V_0 T(g,\beta_c)\cos\varphi_{c,n} + \frac{V_0}{2}\frac{dS}{dg}\cdot(g_{n,1} - g_{n,2})\sin\varphi_c. \quad (21)$$

Again, since T and S are slowly varying functions, equations (19) and (21) approximate to the usual equations for phase and energy.

Programs have been written based on the above two methods, called LINAC 1 and LINAC 2, and the results compared for a design of the same 20 MeV linac.

Execution times were about 30 secs (CDC 3400) and 9 secs (Atlas). Computed lengths agreed to 10^{-4} cm at low energy, and to 10^{-2} cm at 20 MeV. and design output energy agreed to 0.01 MeV. The motion of a test particle of input phase and energy equal to that of the reference particle in both cases remained within 0.1° of the design value, and, additionally, the approximated formulae (18) and (21) without the second term in LINAC 2 gave phases within 0.5° of the reference value; and in all cases output energy agreed exactly with the design values. The agreement between the two methods is satisfactory, with LINAC 2 having the advantage of more general application; the approximated formulae are acceptable for computation to 0.5°, and have the advantage of simplicity and, again, wider application.

Acknowledgements

The programs mentioned above were written at CERN with much assistance from Mme Y. Marti, and the computations relative to Fig. 3 were made by Mrs. M. Bell; to them the authors express their thanks.

References

1. A. Carne, P.M. Lapostolle, M. Promé, "Accurate Beam Dynamics Equations in Proton Linear Accelerators", V^th Int. Conf. on High Energy Accelerators, Frascati, September 1965.
2. P.M. Lapostolle, "Equations de la Dynamique des Particules dans un Accelerateur Lineaire à Protons", CERN AR/Int. SG/65-11 May 1965.
3. R. Taylor, "Calculation of Drift Tube Dimensions for the Linac Injector for Nimrod", A.E.R.E. - R 3012, 1959.

DISCUSSION

A. CARNE, RHEL

SWENSON, LASL: You must have a condition for the resonance of the entire cell. Do you base this on studies that you have made of the resonant properties of the half cells?

CARNE: Well, if you have a gap which is symmetrical, you have a resonance condition as a function of L upon λ. If the two half cells are different, then you must use separate equations for the two half cells. The first method we are using asks for a cell gap which we can move around within the cell by some small amount, and we assume that this doesn't change the frequency, which is fair. In the second method, we are fixing the phase center and then moving the cell around it. And, because these gaps are different, the two fields are different in each half gap, although the over-all voltages are the same. It turns out in the case that we considered that there was something of the order of a 2 to 3 percent field imbalance between the two half gaps. Again we think this has a very small, or no effect, on frequency, because what you gain in one half, you lose in the second half; so things are going to average out.

SWENSON: Yes. I would like to say that we do essentially the same thing in the PARMILA program, as a result of modifications made during the last year.

OHNUMA, Yale: We have done a quite similar thing at Brookhaven. The difficulty we had is, to what extent we can trust the MESSYMESH calculation on T, S, T′, and S′. You mentioned an approximation for the T and S. How good is this approximation for T′ and S′, which are necessary for orbit calculations?

CARNE: We found that the accuracy for T′ and S′ is not so good. In fact, we found up to 5% for T′ and up to 10% for S′. But T′ and S′ are really quite small corrections, so their accuracy is not so important.

OHNUMA: That is quite true if the velocities of

all the particles are quite similar. However, if you want to follow, for example, a particle which is already unstable, the velocity is quite different from the synchronous velocity, and I don't think T′ is so trivial.

CARNE: Well, we have, of course, designed a linac based on the reference particle. In this paper we have not considered the relation of other particles.

LAPOSTOLLE, CERN: This program was made mainly to design a linac. It is clear that to compute the dynamics of many particles, several things may be changed. In particular, having fixed the dimensions of the cells, one could use for T, T′, S, S′ values which correspond to the actual field distribution. But nevertheless, the program described here is already good for the reference particle.

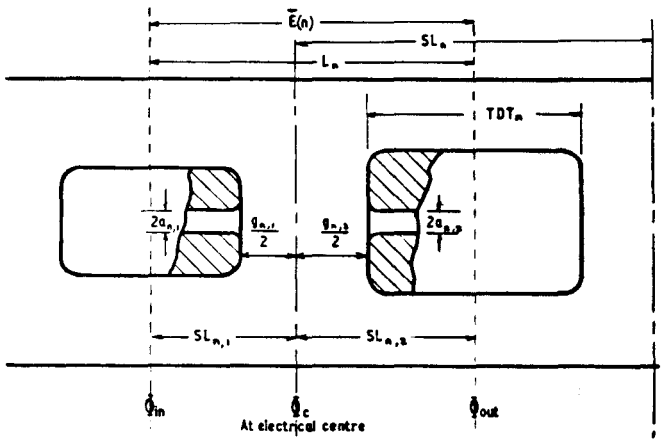

Fig. 1. General cell, n, of a linac.

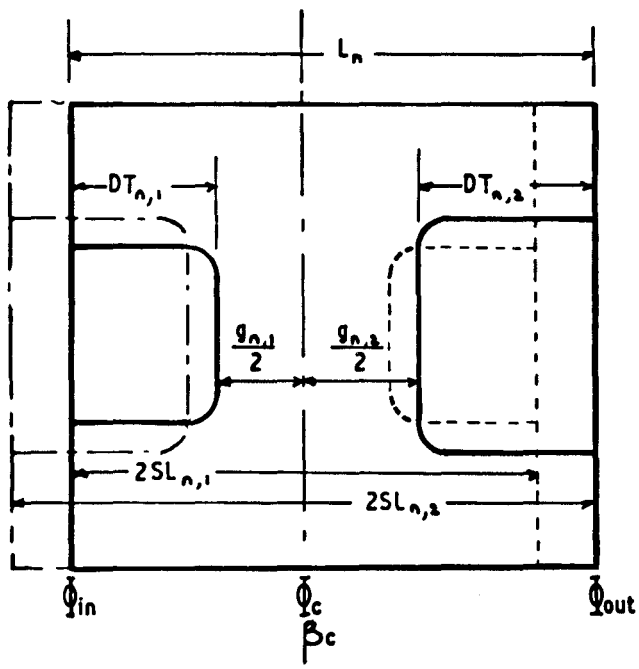

Fig. 2. General cell compounded from two symmetrical cells.

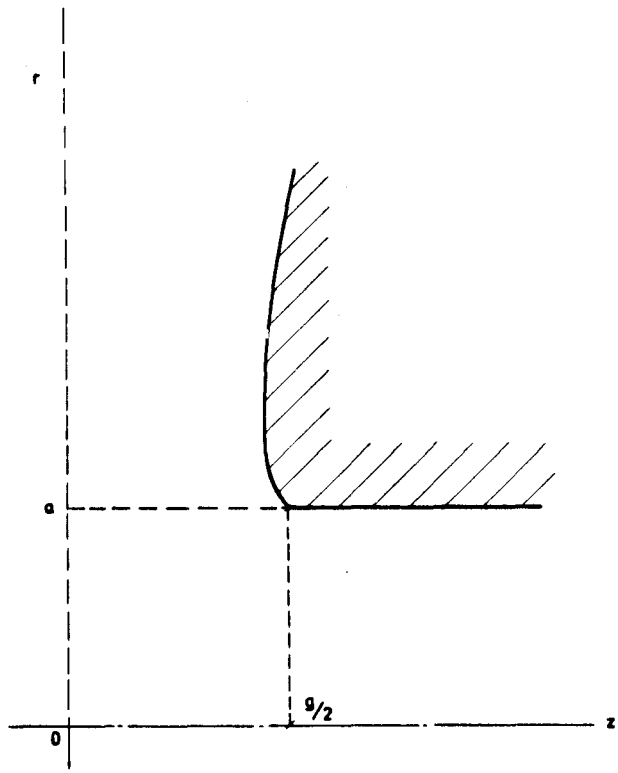

Fig. 3. Uniform field gap profile.