ON A UNIFORM CHARGE DENSITY OF A BEAM AND ITS PHASE-SPACE DISTRIBUTION*

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I. Summary

It is shown that, if a particle beam with a constant velocity and a constant current going through a linear focusing field has a uniform charge density everywhere, its particle distribution in the four-dimensional (transverse) phase space must be of the type assumed by Kapchinskij and Vladimirskij.[1] All the particles are then distributed uniformly on the surface of a four-dimensional ellipsoid. Since the Fourier transform of the Kapchinskij-Vladimirskij distribution has the same form for accelerated and unaccelerated beams, it is expected that this is true for accelerated beams also provided one can ignore the effect of the transverse motion on the longitudinal motion. The method used here is based on the Fourier transform of the particle distribution.[2]

II. Kapchinskij-Vladimirskij Distribution[1]

When a beam of non-relativistic particles going through a linear focusing field has a uniform charge density, the space-charge field is also linear and the equations of transverse motions are

$$\ddot{x}\ (\equiv d^2x/dt^2) = -K_x(t)\ x\ , \tag{1}$$

$$\ddot{y} = -K_y(t)\ y\ . \tag{2}$$

It is assumed here that the effect of the transverse motion on the longitudinal motion can be ignored so that K_x and K_y are functions of time $\underline{t}$ only. An integral of the motion is

$$F \equiv (x/\delta_x)^2 + (\delta_x\dot{x} - \dot{\delta}_x x)^2 + (y/\delta_y)^2 + (\delta_y\dot{y} - \dot{\delta}_y y)^2 \tag{3}$$

where $\delta_x(t)$ and $\delta_y(t)$ satisfy the equations of the beam envelopes[3]

$$\ddot{\delta}_{x,y} + K_{x,y}\ \delta_{x,y} - 1/\delta^3_{x,y} = 0\ . \tag{4}$$

Kapchinskij and Vladimirskij,[1] in their work on limitations of proton beam current in a linear accelerator, assumed the particle distribution in the transverse phase space

$$n(x,y,\dot{x},\dot{y}) = n_o\ \delta(F-F_o) \tag{5}$$

where n_o and F_o are constants. All particles are then distributed uniformly on the surface of a four-dimensional ellipsoid, $F = F_o$. The space charge density over the elliptic beam cross section, $(x/\delta_x)^2 + (y/\delta_y)^2 = F_o$,

$$\rho(x,y) = \int_{-\infty}^{\infty}\int n_o\ \delta(F-F_o)\ d\dot{x}d\dot{y} = \pi n_o/\delta_x\delta_y \tag{6}$$

is everywhere independent of x and y. Charge densities in the projected ellipses

$$(x/\delta_x)^2 + (\delta_x\dot{x} - \dot{\delta}_x x)^2 = F_o\ , \tag{7}$$

$$(y/\delta_y)^2 + (\delta_y\dot{y} - \dot{\delta}_y y)^2 = F_o\ . \tag{8}$$

in $(x,\dot{x})$ and $(y,\dot{y})$ phase spaces are also uniform:

$$\rho(x,\dot{x}) = \rho(y,\dot{y}) = \pi n_o\ . \tag{9}$$

The current in the beam is

$$I = \pi v\ r_x r_y\ \rho(x,y) \tag{10}$$

where $\underline{v}$ is the velocity (common to all particles) and

$$r_{x,y}(t) = \sqrt{F_o}\ \delta_{x,y}(t) \tag{11}$$

are maximum beam excursions in x- and y-directions, respectively.

The Fourier transform[2] of the distribution, Eq. (5),

$$N(s,u,v,w) = (1/4\pi^2)\iiiint_{-\infty}^{\infty} dxdyd\dot{x}d\dot{y}\ n(x,y,\dot{x},\dot{y}) \times \exp\{i(sx+uy+v\dot{x}+w\dot{y})\} \tag{12}$$

can be expressed in terms of Bessel function J_1 as follows:

$$N(s,u,v,w) = (n_oF_o/2)\ J_1(f)/f \tag{13}$$

where

$$f \equiv \{F_o(s'^2 + u'^2 + v'^2 + w'^2)\}^{1/2} \tag{14}$$

and

$$\begin{bmatrix} s' \\ u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \delta_x & 0 & \dot{\delta}_x & 0 \\ 0 & \delta_y & 0 & \dot{\delta}_y \\ 0 & 0 & \frac{1}{\delta_x} & 0 \\ 0 & 0 & 0 & \frac{1}{\delta_y} \end{bmatrix} \begin{bmatrix} s \\ u \\ v \\ w \end{bmatrix}\ . \tag{15}$$

In deriving Eq. (13), it is convenient to transform the integral (12) into an integral over polar coordinates and use the relation[4]

$$\int_0^{\pi/2} d\alpha \sin\alpha \cos\alpha\, J_o(a\cos\alpha)\, J_o(b\sin\alpha)$$

$$= J_1(\sqrt{a^2+b^2})/\sqrt{a^2+b^2}\ . \qquad (16)$$

III. Fourier Transform of a Uniform Charge Distribution

It has been shown that the Kapchinskij-Vladimirskij distribution in phase space leads to a uniform charge density in (xy) space. The question is: What is the most general form of the phase space distribution that gives a uniform charge density everywhere. A partial answer to this has been obtained by Walsh[2] and his result is given below in a slightly modified form.

One starts with a phase space distribution of particles at $t = t_o$,

$$n_o(x_o, y_o, \dot{x}_o, \dot{y}_o)$$

and requires that the corresponding charge density is uniform over an elliptic beam cross section,

$$\rho(x,y) = \rho_o \quad \text{for } \{x/\delta_x(0)\}^2 + \{y_o/\delta_y(0)\}^2 \le F_o$$

$$= 0 \quad \text{otherwise.} \qquad (17)$$

The Fourier transform $N_o(s,u,v,w)$ of $n_o(x_o,y_o,\dot{x}_o,\dot{y}_o)$ must then satisfy the condition

$$N_o(s,u,0,0) = (1/2\pi)\rho_o\delta_x(0)\delta_y(0)F_oJ_1(g_o)/g_o \qquad (18)$$

where

$$g_o \equiv \sqrt{F_o[\{\delta_x(0)s\}^2 + \{\delta_y(0)u\}^2]}\ . \qquad (19)$$

At $t = t_1$, the distribution is

$$n_1(x_1,y_1,\dot{x}_1,\dot{y}_1) = n_o(x_o,y_o,\dot{x}_o,\dot{y}_o)$$

$$\times\ \partial(x_o,y_o,\dot{x}_o,\dot{y}_o)/\partial(x_1,y_1,\dot{x}_1,\dot{y}_1) \qquad (20)$$

Since the system is linear and the charge density is assumed to be uniform everywhere, one can write

$$\begin{bmatrix} x_o \\ \dot{x}_o \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} ; \begin{bmatrix} y_o \\ \dot{y}_o \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ \dot{y}_1 \end{bmatrix} \qquad (21)$$

and $\alpha_{11}\ \alpha_{22} - \alpha_{12}\ \alpha_{21} = \beta_{11}\ \beta_{22} - \beta_{12}\ \beta_{21} = 1$. α's and β's are functions of t_o and t_1 only.

The Fourier transform N_1 of n_1 is related to N_o by

$$N_1(s_1,u_1,v_1,w_1) = N_o(\alpha_{22}s_1 - \alpha_{21}v_1, \beta_{22}u_1 - \beta_{21}w_1, -\alpha_{12}s_1 + \alpha_{11}v_1, -\beta_{12}u_1+\beta_{11}w_1) \qquad (22)$$

On the other hand, analogous to Eq. (18), one can write

$$N_1(s_1,u_1,0,0) = (1/2\pi)\rho_1\delta_x(1)\delta_y(1)F_oJ_1(g_1)/g_1 \qquad (23)$$

with

$$g_1 \equiv \sqrt{F_o[\{\delta_x(1)s_1\}^2 + \{\delta_y(1)u_1\}^2]}\ . \qquad (24)$$

The beam shape at $t = t_1$ in (x_1y_1) space is

$$\{x_1/\delta_x(1)\}^2 + \{y_1/\delta_y(1)\}^2 = F_o \qquad (25)$$

and the uniform charge density is ρ_1 within this ellipse. If the beam is <u>unaccelerated</u> from $t = t_o$ to t_1 and the current is constant, one obtains from Eqs. (10) and (11)

$$\rho_o\ \delta_x(0)\ \delta_y(0) = \rho_1\ \delta_x(1)\ \delta_y(1) \qquad (26)$$

so that

$$N_o(\alpha_{22}s_1, \beta_{22}u_1, -\alpha_{12}s_1, -\beta_{12}u_1) = N_1(s_1,u_1,0,0)$$

$$= (1/2\pi)\ \rho_o\ \delta_x(0)\ \delta_y(0)\ F_o\ J_1(g_1)/g_1 \qquad (27)$$

If one makes the substitution

$$s = \alpha_{22}s_1,\quad u = \beta_{22}u_1,\quad v = -\alpha_{12}s_1,\quad w = -\beta_{12}u_1, \qquad (28)$$

one gets, from Eqs. (15), (17), (21), and (25),

$$\{\delta_x(1)s_1\}^2 + \{\delta_y(1)u_1\}^2 = s'^2+u'^2+v'^2+w'^2 \qquad (29)$$

where (s',u',v',w') are given by Eq. (15) with $\delta_x = \delta_x(0)$, etc. The Fourier transform N at an arbitrary time t is then

$$N(s,u,v,w) = (1/2\pi)\rho(t)\delta_x(t)\delta_y(t)F_oJ_1(f)/f \qquad (30)$$

where f is given by Eq. (14) and this is identical to the Fourier transform, Eq. (13), of the Kapchinskij-Vladimirskij distribution.

IV. Inverse Transformation

The remaining question is to find the phase space distribution $n(x,y,\dot{x},\dot{y})$ starting from the Fourier transform, Eq. (30), by a Fourier inversion,

$$n(x,y,\dot{x},\dot{y}) = (1/4\pi^2) \iiiint_{-\infty}^{\infty} ds\,du\,dv\,dw\ N(s,u,v,w)$$

$$\times\ \exp\{-i(sx+uy+v\dot{x}+w\dot{y})\}\ . \qquad (31)$$

By a change of integration variables

$$\begin{bmatrix} S \\ U \\ V \\ W \end{bmatrix} = \sqrt{F_o}\begin{bmatrix} s' \\ u' \\ v' \\ w' \end{bmatrix} \quad ; \quad \begin{bmatrix} s \\ u \\ v \\ w \end{bmatrix} = 1/\sqrt{F_o}\begin{bmatrix} \frac{1}{\delta_x} & 0 & -\dot{\delta}_x & 0 \\ 0 & \frac{1}{\delta_y} & 0 & -\dot{\delta}_y \\ 0 & 0 & \delta_x & 0 \\ 0 & 0 & 0 & \delta_y \end{bmatrix}\begin{bmatrix} S \\ U \\ V \\ W \end{bmatrix} \tag{32}$$

one can write

$$n(x,y,\dot{x},\dot{y}) = (1/4\pi^2)(1/2\pi)\rho\delta_x\delta_y(1/F_o)$$

$$\iiiint dSdUdVdW\,\{J_1(A)/A\}$$

$$\times \exp\{-i(Sx'+Uy'+V\dot{x}'+W\dot{y}')\} \ . \tag{33}$$

where

$$A \equiv S^2 + U^2 + V^2 + W^2 \ , \tag{34}$$

$$\begin{bmatrix} x' \\ y' \\ \dot{x}' \\ \dot{y}' \end{bmatrix} = 1/\sqrt{F_o}\begin{bmatrix} 1/\delta_x & 0 & 0 & 0 \\ 0 & 1/\delta_y & 0 & 0 \\ -\dot{\delta}_x & 0 & \delta_x & 0 \\ 0 & -\dot{\delta}_y & 0 & \delta_y \end{bmatrix}\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} . \tag{35}$$

Note that the integral of the motion F defined by Eq. (3) can be written as

$$F = F_o(x'^2 + y'^2 + \dot{x}'^2 + \dot{y}'^2) \ . \tag{36}$$

Using polar coordinates $(T,\alpha,\theta,\varphi)$,

$$\begin{aligned} S &= T\cos\alpha\cos\theta \ ; & dSdUdVdW \\ U &= T\cos\alpha\sin\theta & = (\tfrac{1}{2})T^3\sin(2\alpha)dTd\alpha d\theta d\varphi \\ V &= T\sin\alpha\cos\varphi & \iiiint_{-\infty}^{\infty} \rightarrow \\ W &= T\sin\alpha\sin\varphi & \end{aligned}$$

$$\int_0^\infty dT \int_0^{\pi/2} d\alpha \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi$$

one can reduce the integral (33) to

$$n(x,y,\dot{x},\dot{y}) = (\rho\delta_y\delta_y/\pi F_o)(1/2k)$$

$$\times \int_0^\infty dT\,\{TJ_1(T)\,J_1(kT)\} \tag{37}$$

where $k^2 = F/F_o$. The integral in Eq. (37) is performed in Appendix and the final result is

$$n(x,y,\dot{x},\dot{y}) = (\rho\delta_x\delta_y/\pi F_o)\,\delta(k^2-1)$$

$$= (\rho\delta_x\delta_y/\pi)\,\delta(F-F_o) \ . \tag{38}$$

From Eqs. (5) and (6), one sees that this is obviously the Kapchinskij-Vladimirskij distribution. Since the Fourier transform, Eq. (13), of the Kapchinskij-Vladimirskij distribution, which is identical to the Fourier transform, Eq. (30), for unaccelerated beams, applies to an accelerated beam also, it is expected that the Kapchinskij-Vladimirskij distribution is in general the only possibility corresponding to a uniform charge density.

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References

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4. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge, 1958, p. 376.
5. G. N. Watson, loc. cit., p. 134.
6. For example, L. I. Schiff, Quantum Mechanics, McGraw-Hill, New York, 1949, p. 51.

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Appendix

The integral in Eq. (37) can be written as[5]

$$I(k) \equiv (1/2k)\int_0^\infty dT\,\{TJ_1(T)J_1(kT)\}$$

$$= \lim_{M\to\infty}(1/2k)\int_0^M dT\,\{TJ_1(T)J_1(kT)\}$$

$$= \lim_{M\to\infty}(1/2k)\left\{\frac{kMJ_2(kM)J_1(M)-MJ_1(kM)J_2(M)}{k^2-1}\right\}$$

$$= \lim_{M\to\infty}(1/2k\sqrt{k})\left\{\frac{\sin(k-1)M}{\pi(k-1)} + \frac{\cos(k+1)M}{\pi(k+1)}\right\} . \tag{A1}$$

Since $k = F/F_o \geq 0$, the second term in the bracket gives no contribution to integrals of the form

$$\int_0^\infty I(k)\, f(k)\, dk$$

when $M \to \infty$. On the other hand, the first term is a well-known expression of the δ-function,[6]

$$\lim_{M \to \infty} \frac{\sin (k-1) M}{\pi(k-1)} = \delta(k-1) \qquad (A2)$$

so that

$$I(k) = (1/2k\sqrt{k})\ \delta(k-1) = (1/2)\ \delta(k-1). \qquad (A3)$$

Since[6]

$$\delta(k^2-1) = (1/2)\ \{\delta(k-1) + \delta(k+1)\}$$
$$= (1/2)\ \delta(k-1) \text{ for } k \geq 0, \qquad (A4)$$

one finally gets

$$I(k) = \delta(k^2-1) = \delta(F/F_o - 1)$$
$$= F_o\ \delta(F - F_o)\ . \qquad (A5)$$

DISCUSSION

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LAPOSTOLLE, CERN: I have two comments. One is about this question of phase space uniform distribution or Kapchinskij - Vladimirskij distribution. In case one restricts oneself to hyperspherical or hyperellipsoidal distributions, the density only depends on the "distance" from the center. One can easily find that between a two-dimensional and a four-dimensional distribution, one is just the derivative of the other in such a way that in order to have a uniform distribution in the two-dimensional one, the four-dimensional distribution must be a delta function at the edge, that is to say, a surface distribution. But if you use a different distribution, you can easily relate it to the four-dimensional distribution. For instance, a Gaussian distribution remains Gaussian, and this is closer to an actual distribution. If one goes to a six-dimensional distribution and relates the two-dimensional distribution to it, one finds that one is the second derivative of the other, so that in order to get a uniform distribution in two dimensions, one has to have a double layer distribution in six dimensions, which is highly improbable. But again the second derivative relation can be used to compute any kind of distribution provided, again, it is of hyperellipsoidal shape. Now, concerning transverse phase space dilution, I might mention that computations made in Saclay for their linac design, and making use of programs which will be described tomorrow, also gave a factor of about 2 in phase space area between input and output.

OHNUMA: I would like to make one comment: I wanted to get a relation between phase space and ordinary space for a general distribution, but the difficulty here is, of course, that for the general nonuniform distribution, the transformation in the phase space is not linear. So it becomes rather difficult to say anything general.