BEAM LOADING AND FREQUENCY DETUNING IN A LINEAR ACCELERATOR

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Abstract

We model the interaction of a particle beam with the electromagnetic field corresponding to that of the Alvarez drift-tube linac type structures. The motivation of the research is to discover the influence of the particle beam on the Q of the cavity and further, to uncover the detuning of the resonance frequency of the cavity due to the loading of the cavity by the particle beam.

The method employed involves the concept of "complex" frequency detuning of the cold, or unperturbed, cavity resonance frequency. The real component of the frequency detuning corresponds to the observable shift in the cavity resonance frequency and the imaginary component of frequency detuning is related to the Q of the cavity. The formalism of frequency detuning is based upon a field function analysis used in microwave tubes. In the analysis presented the frequency detuning is shown to be intimately related to the energy transfer between the particle beam and the electromagnetic field within the cavity. The energy transfer is evaluated using a particle dynamics technique.

Introduction

A linear accelerator cavity has a set of characteristic eigenfrequencies and eigenmodes. The introduction of an energetic, heavy, charged particle beam, such as a beam of protons, perturbs them from their initial "cold" values. A field with an initial harmonic time dependence of $E_\nu = E_o \exp(j\omega_\nu t)$, (where ν indicates a particular mode) in the presence of a particle beam becomes $E_\nu' = E_o \exp(j(\omega_\nu + \delta\omega_\nu)t)$, where $\delta\omega_\nu (= \delta\omega_{\nu r} + j\omega_{\nu i})$ represents a complex frequency shift. The real part of the complex frequency shift, $\delta\omega_{\nu r}$, represents a frequency detuning or pulling in the resonance frequency of the cavity.

In this article, the relationship between the real frequency detuning and the imaginary, out of phase, power flow is developed. The imaginary component of the complex frequency detuning, $\delta\omega_{\nu i}$, is proportional to the rate at which the particles exchange energy with the microwave fields. In order that the particles gain energy it is required that $\delta\omega_{\nu i}$ remain a positive quantity. The relationship between the frequency detuning and the power transferred between the beam and a cavity field is analyzed using a distributed constant or field function analysis. Alternatively, the system may be analyzed using a "lumped constant" approach in which the cold cavity is represented by a resonant inductor and capacitor L-C circuit and the effect of the beam is represented by an electronic admittance. Herein, both the field function and lumped constant approach are pursued.

Field Function Analysis of Frequency Detuning

When charged particles are injected into cavities or systems with distributed constants, the resulting changes in frequency may be expressed in terms of the electromagnetic field and the forced convection current. This is achieved by means of a perturbation method. The problem may be viewed as an adiabatic change in the ratio of the cold cavity energy to the cold cavity resonant energy (where cold indicates the absence of a charged particle beam), such that this ratio remains constant for a small perturbation. An adiabatic change is defined as a perturbation which does not cause a change from the cold mode to a neighboring mode.[1] Under this constraint, the frequency shift may be obtained using a quantum mechanical analysis or a general field function analysis. The latter proves the more instructive.

The field function perturbation approach will now be detailed. When charged particles are injected into the Linac cavity, the cold field distribution changes and the resonant frequency will also change. The field in the cavity will act on the particles, and thus the particle flux, will vary with time and position in the cavity. This modified current, in turn, modifies the field, and this process continued as the particles traverse the accelerating gaps. However, for a short transit time between the gaps a good approximation of the charged particle current density within the cavity is obtained by calculating the particle velocities that are produced from the interaction with the cold, unperturbed electromagnetic field within the cavity. The fields within the cavity may now be expanded as a summation of the cold cavity modes

$$\left.\begin{aligned} \mathbf{E} &= \sum_\nu a_\nu(t)\mathbf{e}_\nu(x,y,z) \\ \mathbf{H} &= \sum_\nu b_\nu(t)_\nu(x,y,z) \end{aligned}\right\} . \tag{1}$$

Substitution of the above modal expression in the following identity

$$\int_{S,S'} (\mathbf{A} \times \mathbf{B}^*)\cdot\hat{\mathbf{n}}\, dS = \int_V \mathbf{B}^*\cdot\nabla\times\mathbf{A}\, dV - \int_V \mathbf{A}\cdot\nabla\times\mathbf{B}^*\, dV \tag{2}$$

where the left-hand side is integrated over the surfaces S, such that the tangential component of $\mathbf{e}_\nu$ is zero, and S´, such that the tangential component of $\mathbf{h}_\nu$ is zero, gives:

$$\int_{S'} (\hat{\mathbf{n}} \times \mathbf{H}^*)\cdot\mathbf{e}_\nu\, dS = \varepsilon_o \frac{da_\nu^*}{dt} + \int_V \mathbf{J}^*\cdot\mathbf{e}_\nu\, dV - k_\nu b_\nu^* \tag{3}$$

$$\int_S (\hat{\mathbf{n}} \times \mathbf{E}^*)\cdot\mathbf{h}_\nu dS = -\mu_o \frac{db_\nu^*}{dt} - k_\nu a_\nu^* \tag{4}$$

where $k_\nu = \omega_\nu/c$ and ε_o, μ_o is the permittivity and permeability of free space. In Eqs. (3) and (4), use has been made of the properties of the field functions detailed in the Appendix. Differentiating Eq. (3) with respect to time t and substituting Eq. (4), we get

$$\frac{1}{c^2}\frac{d^2 a_\nu}{dt^2} + k_\nu^2 a_\nu = -\mu_o \frac{d}{dt}\left[\int_V \mathbf{J}\cdot\mathbf{e}_\nu^* - \int_{S'} (\hat{\mathbf{n}} \times \mathbf{H})\cdot\mathbf{e}_\nu^*\, dS\right] \tag{5}$$

$$- k_\nu \int_S (\hat{\mathbf{n}} \times \mathbf{E})\cdot\mathbf{h}_\nu^*\, dS$$

where the complex conjugate has been taken. The third integral on the right-hand side of Eq. (5) is integrated over the surface S for which $\hat{\mathbf{n}} \times \mathbf{h}_\nu^* \neq 0$, i.e., it represents ohmic losses in the walls of the cavity. Making use of the relation[1]

$$\hat{\mathbf{n}} \times \mathbf{E} = (1 + j)\frac{\mathbf{H}}{\delta\sigma}$$

where δ is the skin depth and σ the conductivity of the cavity, allows Eq. (5) to be rewritten as:

$$(d^2/dt^2 + \omega_\nu^2)a_\nu = -\mu_o c^2 d/dt\left[\int_V \mathbf{J}\cdot\mathbf{e}_\nu^* dV - \int_{S'} (\hat{\mathbf{n}} \times \mathbf{H})\cdot\mathbf{e}_\nu^* dS\right]$$

$$+ (1 - j)\,\omega_\nu^2 a_\nu/Q_o \tag{5'}$$

where Q_o is the ohmic quality factor of the cavity. It is assumed that little intermode coupling takes place in the walls of the cavity. Multiplying this equation by a_ν^* and assuming an harmonic time dependence of the form exp(jωt) gives:

$$\omega^2 - \omega_\nu^2(1 - \frac{1-j}{Q_o}) = \frac{j\omega}{U}(P - \frac{1}{2}\int_{S'}(\hat{\mathbf{n}} \times \mathbf{H})\cdot \mathbf{E}_\nu^* dS) \qquad (6)$$

Here, the power transfer is given by $P = 1/2\int_V \mathbf{J}\cdot\mathbf{E}^* dV$ and the stored energy by $U = \frac{1}{2}\epsilon_o\int_V \mathbf{E}\cdot\mathbf{E}^* dV$. The remaining integral on the right-hand side of Eq. (6) is non-zero over the surfaces of the microwave feed apertures.[2] Expanding ω in terms of the cold value, ω_ν, and a small perturbation, $\delta\omega_\nu$, gives

$$\delta\omega_{\nu r}\left[2 - \frac{1-j}{Q_o}\right] = j\frac{P}{U} - \frac{j}{2U}\int_{S'}(\hat{\mathbf{n}} \times \mathbf{H})\cdot \mathbf{E}_\nu^* dS - \omega_\nu(1-j)/Q_o \ . \qquad (7)$$

For large Q values this equation reduces to

$$\left.\begin{aligned} \delta\omega_{\nu r} &= -\frac{1}{2}P_i/U \\ \delta\omega_{\nu i} &= \frac{1}{2}P_r/U \end{aligned}\right\} \qquad (8)$$

where the subscripts r and i refer to the real and imaginary components of the complex quantities. From this equation, one can readily identify the real, or physically measurable, frequency detuning to be directly proportional to the out of phase rate energy transfer to the particle beam. Further, the imaginary component of frequency detuning represents a wave damping and therefore one can define a quality factor due to beam loading as $Q_b = \omega_\nu/2\delta\omega_{\nu i} = \omega_\nu U/P_r$. For a given beam quality factor, Q_b, the frequency detuning can be written as:

$$\delta\omega_{\nu r}/\omega_\nu = -(2Q_b)^{-1}(P_i/P_r). \qquad (9)$$

Thus, the frequency detuning is determined from the ratio of the imaginary to the real component of power transfer.

It now remains to evaluate the power transfer in terms of parameters appropriate to a phase synchronous charged particle beam. This is discussed in the following section.

Particle Dynamics

On entering an accelerating gap, charged particles are assumed to encounter an electric field of the form $E(t) = \mathrm{Re}\{E e^{j\omega t}\}$, where $E = -jE_o$. For heavy charged particles, such as protons, the change in particle mass, M, with particle velocity is small; therefore, it is appropriate to solve the non-relativistic Lorentz equation for the particle velocity v. The particle velocity is written in the form $v = v_o + \delta v$, where δv is due to the interaction with the accelerating field, and v_o is the drift velocity of a particle along the axis of the accelerator. The equation of motion is:

$$m\frac{d}{dt}\delta v = qE(t) = qE e^{j\omega t} \ . \qquad (10)$$

Thus,

$$\delta v = \frac{qE}{m}\left[\frac{e^{j\omega\tau} - 1}{j\omega}\right]e^{j\omega t_1} \ , \qquad (11)$$

where q is the charge of a particle, τ the transit time through an accelerating gap, and t_1 is the time of entry of a particle to a gap region. The work done by the accelerating field on a particle per unit time, i.e., the power transfer, is given by:

$$p = \frac{q}{2}vE^*(t) = \frac{1}{2}q(v_o + \delta v)E^*(t)$$
$$= \frac{1}{2}qv_oE^*e^{-j\omega t_1}e^{-j\omega\tau} + \frac{1}{2}q^2\frac{|E|^2}{m}\left[\frac{1-e^{-j\omega\tau}}{j\omega}\right] \ . \qquad (12)$$

Summing over all particles corresponds to integrating $n_o f(v_o)p$ over velocity and space (where f is the initial velocity distribution and n_o is the initial line density). The distribution function is normalized according to $n_o\int_{-\infty}^{\infty} f(v_o)\,dv_o = n_o$. For a monoenergetic velocity distribution and changing the spatial integration to an integration over the transit time τ gives:

$$P = P_{01}\left[\frac{1-j\theta-e^{-j\theta}}{2\theta^2}\right] + P_{02}\left[\frac{1-e^{-j\theta}}{2\theta}\right]e^{-j\phi_s} \qquad (13)$$

where

$$P_{01} = I_{c1}qv_o\tau^2E_0^2/m, \quad P_{02} = I_{c1}g\frac{\sin(\delta\phi_s/2)}{(\delta\phi_s/2)}E_o \ ,$$

I_{c1} is the d.c. current, $\theta = \omega\tau$, g is the gap length, ϕ_s the synchronous phase, $\pm\delta\phi_s/2$ the deviation from the synchronous phase. In Eq. (13) it is assumed that $d\tau \approx v_o\,dz$ and the power transfer has been averaged with respect to the initial injection phase. The real and imaginary components of power transfer are given by:

$$P_r = \frac{P_{01}}{4}\cdot\frac{\sin^2(\theta/2)}{(\theta/2)^2} - \mathrm{Im}(P_{02}e^{-j\phi_s})\frac{\sin\theta}{2\theta} + \mathrm{Re}(P_{02}e^{-j\phi_s})\frac{\sin^2(\theta/2)}{\theta} \qquad (14)$$

$$P_i = \frac{P_{01}}{2}\left(\frac{\sin\theta - \theta}{\theta^2}\right) + \mathrm{Im}(P_{02}e^{-j\phi_s})\frac{\sin^2(\theta/2)}{\theta} + \mathrm{Re}(P_{02}e^{-j\phi_s})\frac{\sin\theta}{2\theta} \ . \qquad (15)$$

The frequency detuning induced by the beam follows from Eq. (8).

It is interesting to discover the relative influence of the two terms in Eq. (13). To this end, let $\theta \to 0$ and define $V_o = mv_o^2/(2q)$ and $V_{r.f} = Eg$. This enables the power transfer to be evaluated as

$$\lim_{\theta\to 0}\{P\} = \frac{1}{4}P_{01}(1 + 2(P_{02}/P_{01})je^{-j\phi_s}) \qquad (16)$$

where

$$|P_{02}/P_{01}| = |(2V_o/V_{r.f})\sin(\delta\phi_s/2)/(\delta\phi_s/2)| >> 1 \ .$$

Thus, the second term in Eq. (13), i.e., that dependent on the synchronous phase, is the dominant term.

The frequency detuning by the particle beam may be analyzed from a lumped element perspective and this is detailed in the following section.

Lumped Element Analysis of Frequency Detuning

From the previous section the power transfer is given by $P = 1/2\int qn_o(v_o + \delta v)E^* dz$. In the lumped element equivalent circuit approach we define $E_o = V_{r.f}/g$ and thus obtain the current due to the presence of a charged particle beam as:

$$I = \int(qn_o/g)(v_o + \delta v)E^*/E_o \, dz \ . \qquad (17)$$

It is again assumed that $dz \approx v_o\,d\tau$ and Eq. (11) is substituted in Eq. (17), resulting in:

$$I = \frac{I_{c1}}{g}\left[(qE_o\tau^2/m)\left(\frac{1-j\theta-e^{-j\theta}}{\theta^2}\right)\right]$$

$$+ I_{c1}\frac{\sin(\delta\phi_s/2)}{(\delta\phi_s/2)}\left(\frac{1-e^{-j\theta}}{\theta}\right)e^{-j\phi_s} \tag{18}$$

where the current has been averaged with respect to the initial injection phase. Defining $Y_e = I/V_{r.f} = I/(E_{og})$, enables the electronic admittance of the beam to be evaluated as:

$$Y_e = \left(\frac{I_{c1}}{2V_o}\right)[(1-\cos\theta)/\theta^2 - j(\theta - \sin\theta/\theta^2)]$$

$$+ \left(\frac{2I_{c1}}{V_{r.f}}\right)\frac{\sin(\delta\phi_s/2)}{(\delta\phi_s/2)}\left[\cos\phi_s\frac{\sin^2(\theta/2)}{\theta}\right.$$

$$\left. + \sin\phi_s\cdot\frac{\sin\theta}{2\theta} + j\left(\cos\phi_s\frac{\sin\theta}{2\theta} - \sin\phi_s\frac{\sin^2(\theta/2)}{\theta}\right)\right] \tag{19}$$

The accelerating cavity is represented as an inductor and capacitor in parallel loaded by a beam admittance, Y_e. This circuit is shown in Fig. 1, where C_o is defined as the total equivalent capacitance of the system, i.e.,

$$\frac{1}{2}C_o V_{r.f}^2 = \frac{1}{2}\epsilon_o\int_V |E|^2 dV \ . \tag{20}$$

Thus, C_o is representative of the energy stored within the system.

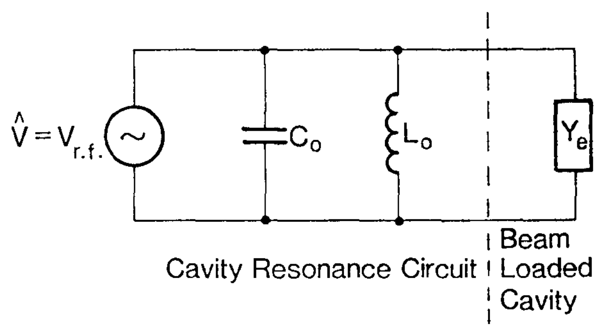

Figure 1. Equivalent circuit of accelerating cavity loaded by a particle beam.

At resonance the total susceptance of the circuit is zero, i.e.,

$$B_e + B_c = 0 \tag{21}$$

Here, B_e represents the imaginary part of Y_e, i.e., the electronic susceptance, and B_c the circuit susceptance is given by:

$$B_c = \frac{C_o}{\omega}(\omega^2 - \omega_o^2) \tag{22}$$

where $\omega_o (= (L_o C_o)^{-1/2})$ is the resonance frequency of the cavity before the beam is introduced. Let the beam introduce a small perturbation $\delta\omega$ about ω_o such that $\delta\omega/\omega_o << 1$. This enables us to approximate Eq. (22) as

$$B_c = 2C_o\,\delta\omega \ . \tag{22'}$$

Substitution of Eqs. (19) and (22') in Eq. (21) enables the frequency detuning to be evaluated as:

$$\delta\omega = \left(\frac{I_{c1}}{4V_o C_o}\right)(\theta - \sin\theta)/\theta^2 + \left(\frac{I_{c1}}{C_o V_{r.f}}\right)\frac{\sin(\delta\phi_s/2)}{(\delta\phi_s/2)}$$

$$\cdot\left[\sin\phi_s\frac{\sin^2(\theta/2)}{\theta} - \cos\phi_s\frac{\sin\theta}{2\theta}\right] \tag{23}$$

and the electronic conductance introduced by the beam is:

$$G_e = \left(\frac{I_{c1}}{2V_o}\right)\left(\frac{1-\cos\theta}{\theta^2}\right) + \left(\frac{2I_{c1}}{V_{r.f}}\right)\frac{\sin(\delta\phi_s/2)}{(\delta\phi_s/2)}$$

$$\cdot\left[\cos\phi_s\frac{\sin^2(\theta/2)}{\theta} + \sin\phi_s\frac{\sin\theta}{2\theta}\right] \ . \tag{24}$$

If we utilize the relations $V_o = mv_o^2/(2q)$ and $C_o = 2U/(E_o^2 g^2)$, it can readily be seen that Eq. (23) corresponds to Eqs. (8) and (15); the latter two equations are a representation of the real frequency detuning induced by the beam and have been derived via a distributed constant approach. Further, the quality factor of the circuit shown in Fig. 1 is given by:

$$Q = \omega_o C_o / G_e \tag{25}$$

and this can be shown to correspond to the quality factor derived via a distributed constant approach, viz, $Q = \omega_\nu U/P_r$. Thus, regardless of whether one views the problem as one involving distributed constants or lumped circuit elements, both approaches do indeed produce consistent results.

Conclusions

The beam loading and frequency detuning of the cavity have been shown to be intimately related to the real and imaginary components of power transferred to the beam. Further, for a given beam quality the frequency detuning has been shown to be directly proportional to the ratio of the imaginary to the real component of power transfer.

The analyses undertaken using a distributed constant and a circuit element approach produce consistent results.

Appendix A: Field Function Properties

The basic properties of the field functions[1,3,4,5] utilized widely within the main text are detailed herein. Within the cavity, volume V enclosed by surfaces S and S′, the electromagnetic field must satisfy Maxwell's equations and the appropriate boundary conditions. The fields are defined by the following modal expansions:

$$\mathbf{E} = \sum_\nu a_\nu \mathbf{e}_\nu \ , \quad \mathbf{H} = \sum_\nu b_\nu \mathbf{h}_\nu \ .$$

The eigenfunctions and eigenvalues $\mathbf{e}_\nu$ and k_ν, in the cavity are defined such that they satisfy the wave equation:

$$\left.\begin{aligned} (\nabla^2 + k_\nu^2)\mathbf{e}_\nu &= 0 \\ \nabla\cdot\mathbf{e}_\nu &= 0 \end{aligned}\right\} \tag{A1}$$

within volume V and further:

$$\hat{\mathbf{n}}\times\mathbf{e}_\nu = 0 \text{ on } S\ , \quad \hat{\mathbf{n}}\times\mathbf{h}_\nu = 0 \text{ on } S' \tag{A2}$$

where $\hat{\mathbf{n}}$ is a unit vector pointing outwards from a particular surface. The relation between the eigenfunction $\mathbf{h}_\nu$ and $\mathbf{e}_\nu$ is given by

$$k_\nu \mathbf{h}_\nu = \nabla\times\mathbf{e}_\nu \ . \tag{A3}$$

Taking the curl of this equation gives:

$$k_\nu \mathbf{e}_\nu = \nabla\times\mathbf{h}_\nu \tag{A4}$$

and a second curl gives:

$$\nabla^2\mathbf{h}_\nu + k_\nu^2\mathbf{h}_\nu = 0 \tag{A5}$$

provided $\nabla\cdot\mathbf{h}_\nu = 0$ within the volume V. Integrating (A3) over the surface S gives:

$$\int_S k_\nu \mathbf{h}_\nu\cdot\hat{\mathbf{n}}\,dS = \int_S (\nabla\times\mathbf{e}_\nu)\cdot\hat{\mathbf{n}}\,dS$$

and the right-hand side of this equation is zero due to Eq. (A2). Therefore,

$$\hat{\mathbf{n}}\cdot\mathbf{h}_\nu = 0 \text{ on } S \text{ and similarly } \hat{\mathbf{n}}\cdot\mathbf{e}_\nu = 0 \text{ on } S' \qquad \text{(A6)}$$

Normalization is chosen such that

$$\int_V \mathbf{h}_\nu\cdot\mathbf{h}_\mu^* dV = \int \mathbf{e}_\nu\cdot\mathbf{e}_\mu^* dV = \delta_{\nu\mu} \qquad \text{(A7)}$$

where $\delta_{\nu\mu}$ is the Kronecker delta function. From these orthogonality relationships one can express $a_\nu(t)$ and $b_\nu(t)$ in terms of the cavity fields:

$$\left.\begin{aligned} a_\nu &= \int_V \mathbf{E}\cdot\mathbf{e}_\nu^* dV \\ b_\nu &= \int_V \mathbf{H}\cdot\mathbf{h}_\nu^* dV \end{aligned}\right\} \qquad \text{(A8)}$$

The total time-averaged energy stored within the cavity is given by:

$$U = \frac{1}{4}\left[\varepsilon_0 \int_V E\cdot E^* dV + \mu_0 \int H\cdot H^* dV\right]$$

$$= \frac{1}{4}\left[\varepsilon_0 \sum_\nu |a_\nu|^2 + \mu_0 \sum_\nu |b_\nu|^2\right]$$

and from Maxwell's equations for a source free cavity it can be shown that $a_\nu = -j\omega\mu_0 b_\nu/k_\nu$ and $b_\nu = j\omega\varepsilon_0 a_\nu/k_\nu$, thus:

$$U = \frac{1}{2}\mu_0 \sum_\nu |b_\nu|^2 \qquad \text{(A9)}$$

Further, the ohmic Q is given by:

$$Q_o = \frac{\omega \sum_\nu \mu_o |b_\nu|^2}{\mathrm{Re}\left[\int (\hat{\mathbf{n}}\times\mathbf{E})\cdot\mathbf{H}^* dS\right]}$$

which, for a skin depth δ, can be rewritten as:

$$Q_o = 2\left(\delta \int |\mathbf{h}_\nu|^2 dS\right)^{-1} \qquad \text{(A10)}$$

References

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