NONLINEAR SPACE CHARGE EFFECTS AND EMITTANCE GROWTH IN LINAC*

Yinbao Chen and Zhibin Huang
China Institute of Atomic Energy, P.O. Box 275(17), Beijing 102413

Abstract

The nonlinear space charge effect of bunched beam in linac is one of the important reasons that induce the emittance growth. The general formulas for calculating the potential of space charge with nonuniform distribution in surrounding structure are presented. For a bunched beam with different distribution in waveguide of linac, the expresses of the nonlinear field energy of a cylinder model of space charge are derived, and the numerical results of the nonlinear field energy for different density distributions are given. The emittance growth caused by these nonuniformities are discussed.

1 INTRODUCTION

In high-current beam for Free Electron Laser (FEL) and linear accelerator for high energy physics, induction linac for heave ion fusion, microwave devices and other applications, the space charge force is no longer small compared with the externally applied focusing forces. And the space charge effect is assumed to be one of the fundamental factors governing the beam dynamics.

Since the theoretical study and numerical simulation show that the nonuniform particle distributions have more electrostatic field energy per unit length than that of the equivalent uniform beam with the same current I, rms radius, and rms emittance. Therefore, it is suggested that this additional field energy is converted into particle kinetic energy and caused emittance growth as the distribution tends to become more homogeneous. This concept has been already accepted by many studies. Historically, the relationship between rms emittance and space charge field energy term for a continuous beam in a continuous focusing channel was firstly derived by Lapostolle [1]. The rms envelope equation with space charge was obtained by Sacherer [2]. An equation for emittance growth in space-charge-dominated beam having nonuniform density was derived by Struckmeier, Klabunde, and Reiser [3]. For a round continuous beam with an arbitrary distribution in a linear focusing channel, a differential equation for emittance change was derived by Wangler [4]. And, a generalized differential equation for a bunched beam was derived by Hofmann and Struckmeier [5]. Also, there are many further study results on the subject (see for examples Ref. [6] and [7]).

We should point it out that the above results concerning the calculation of the space charge field energy are based on the assumption of a continuous beam in a tube or a bunched beam in free space though involved with different density distributions. There is an obvious difference between these results and the reality of the electron (or ion) bunched beam in a linear accelerator or some microwave devices.

With regard to a bunched beam in surrounding structures, the space charge effects of nonuniform density distributions in waveguide of linac have been studied in our early work [8]. In this paper, first we review the main point of Ref. [8] to give the general formulas for potential induced by a cylinder of space charge with nonuniform density distribution in a surrounding cylinder (Section 2); in Section 3, for a bunched beam with nonuniform distributions in waveguide of linac, we present the expresses of the nonlinear field energy of a cylinder model of space charge; in Section 4, we show the numerical results of the nonlinear field energy for different density distributions; and finally, in Section 5, we discuss the emittance growth for a bunched beam in linac.

2 GENERAL FORMULAS FOR POTENTIAL OF NONUNIFORM CHARGE DISTRIBUTION

For the convenience of understanding and application, here we review the main point of Ref.[8] in which the general formulas for calculating the potential of space charge with nonuniform distributions in waveguide have been obtained.

A cylinder model of space charge is used to present a space charge bunch in linac as shown in Fig.1.

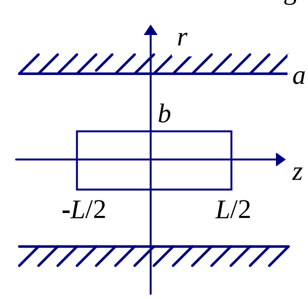

Figure 1. Cylinder model of space charge.

Letting a as the accelerator waveguide radius, b and $\pm L/2$ as the boundaries of the cylinder model in r and z directions,

* Supported by National Natural Science of China.

respectively. The potential induced by the space charge bunch with uniform distribution ρ in a cylindrical coordinate system can be written as follows [9]:

$$\varphi_0 = \rho\, f_0(r,z;b,L/2), \tag{1}$$

where f_0 is the potential induced by a unit space charge density.

According to the general formulas in Ref.[8], for the space charge bunch model with nonuniform charge density distribution $\rho(r,z)=\rho(r)\rho(z)$, we have the induced potential as follows:

$$\varphi = \int_0^{\frac{L}{2}}\int_0^b \rho(\xi,\zeta)\frac{\partial^2 f_0(r,z;\xi,\zeta)}{\partial\xi\partial\zeta}d\xi d\zeta\,. \tag{2}$$

Therefore, so long as the potential φ_0 (in the form of Eq.(1)) induced by a space charge bunch model with uniform distribution is known, the potential φ induced by the space charge with the same model but nonuniform distribution can be obtained from Eq.(2).

The potentials of a cylinder of space charge with uniform density distribution inside a conducting cylinder have been obtained : [9]

$$\varphi_{1,2} = \frac{2\rho ab}{\varepsilon_0}\sum_{l=1}^{\infty}\frac{J_1(k_lb)J_0(k_lr)}{(k_la)^3J_1^2(k_la)}sh\frac{k_lL}{2}e^{-k_l|z|},\quad \left(|z|>\frac{L}{2}\right), \tag{3}$$

$$\varphi_3 = \frac{2\rho ab}{\varepsilon_0}\sum_{l=1}^{\infty}\frac{J_1(k_lb)J_0(k_lr)}{(k_la)^3J_1^2(k_la)}\left[1-ch(k_lz)e^{-\frac{k_lL}{2}}\right],\quad \left(|z|<\frac{L}{2}\right), \tag{4}$$

where $J_i(k_lx)$ is the Bessel function of the ith order, and k_l is chosen so that $J_0(k_la)=0$. Substituting Eqs. (3) and (4) into Eq. (2), we can rewrite the general formulas for calculating the potential induced by the same cylinder of space charge with nonuniform distribution as follows:

$$\varphi_{1,2} = \frac{2}{\varepsilon_0 a}\int_0^{\frac{L}{2}}\int_0^b\rho(\xi,\zeta)\sum_{l=1}^{\infty}\frac{\xi J_0(k_l\xi)J_0(k_lr)}{(k_la)J_1^2(k_la)}ch(k_l\zeta)e^{-k_l|z|}d\xi d\zeta,\quad \left(|z|>\frac{L}{2}\right), \tag{5}$$

$$\varphi_3 = \frac{2}{\varepsilon_0 a}\int_0^{|z|}\int_0^b\rho(\xi,\zeta)\sum_{l=1}^{\infty}\frac{\xi J_0(k_l\xi)J_0(k_lr)}{(k_la)J_1^2(k_la)}ch(k_l\zeta)e^{-k_l|z|}d\xi d\zeta + \frac{2}{\varepsilon_0 a}\int_{|z|}^{\frac{L}{2}}\int_0^b\rho(\xi,\zeta)\sum_{l=1}^{\infty}\frac{\xi J_0(k_l\xi)J_0(k_lr)}{(k_la)J_1^2(k_la)}ch(k_lz)e^{-k_l\zeta}d\xi d\zeta,\quad \left(|z|<\frac{L}{2}\right). \tag{6}$$

Suppose that the density is given by

$$\rho(r,z) = N_{t,l}\rho_{t,l}, \tag{7}$$

where the normalization constants $N_{t,l}$ are chosen to satisfy $2\pi\int_0^b\int_{-\frac{L}{2}}^{\frac{L}{2}}\rho(r,z)rdrdz = Nq$, where N is the total number of particles, the subscription t presents the distribution in transverse direction and l represents that in longitudinal direction. Therefore, t and l can be u (for uniform), w (for waterbag), p (for parabolic) and g (for Gaussian). We list all possible combinations of the four distributions in Table 1, where

$$\rho_{u,u}(r,z)=\frac{Nq}{\pi b^2L},\qquad \rho_{w,u}(r,z)=\frac{2Nq}{\pi b^2L}\left(1-\frac{r^2}{b^2}\right),$$
$$\rho_{p,u}(r,z)=\frac{3Nq}{\pi b^2L}\left(1-\frac{r^2}{b^2}\right)^2,\qquad \rho_{g,u}(r,z)=\frac{Nq}{2\pi\alpha^2L}e^{-\frac{r^2}{2\alpha^2}},$$
$$\rho_{u,w}(r,z)=\frac{3Nq}{2\pi b^2L}\left[1-\frac{z^2}{(L/2)^2}\right],\qquad \rho_{w,w}(r,z)=\frac{3Nq}{\pi b^2L}\left(1-\frac{r^2}{b^2}\right)\left[1-\frac{z^2}{(L/2)^2}\right],$$
$$\rho_{p,w}(r,z)=\frac{9Nq}{2\pi b^2L}\left(1-\frac{r^2}{b^2}\right)^2\left[1-\frac{z^2}{(L/2)^2}\right],\qquad \rho_{g,w}(r,z)=\frac{3Nq}{4\pi\alpha^2L}e^{-\frac{r^2}{2\alpha^2}}\left[1-\frac{z^2}{(L/2)^2}\right],$$
$$\rho_{u,p}(r,z)=\frac{15Nq}{8\pi b^2L}\left[1-\frac{z^2}{(L/2)^2}\right]^2,\qquad \rho_{w,p}(r,z)=\frac{15Nq}{4\pi b^2L}\left(1-\frac{r^2}{b^2}\right)\left[1-\frac{z^2}{(L/2)^2}\right]^2,$$
$$\rho_{p,p}(r,z)=\frac{45Nq}{8\pi b^2L}\left(1-\frac{r^2}{b^2}\right)^2\left[1-\frac{z^2}{(L/2)^2}\right]^2,\qquad \rho_{g,p}(r,z)=\frac{15Nq}{16\pi\alpha^2L}e^{-\frac{r^2}{2\alpha^2}}\left[1-\frac{z^2}{(L/2)^2}\right]^2,$$
$$\rho_{u,g}(r,z)=\frac{Nq}{\pi b^2LD}e^{-\frac{z^2}{2\beta^2}},\qquad \rho_{w,g}(r,z)=\frac{2Nq}{\pi b^2LD}\left(1-\frac{r^2}{b^2}\right)e^{-\frac{z^2}{2\beta^2}},$$
$$\rho_{p,g}(r,z)=\frac{3Nq}{\pi b^2LD}\left(1-\frac{r^2}{b^2}\right)^2e^{-\frac{z^2}{2\beta^2}},\qquad \rho_{g,g}(r,z)=\frac{Nq}{2\pi\alpha^2LD}e^{-\frac{r^2}{2\alpha^2}}e^{-\frac{z^2}{2\beta^2}}, \tag{8}$$

and

$$\alpha^2=\langle r^2\rangle,\quad \beta^2=\langle z^2\rangle,\quad D=\frac{1}{2}\int_{-1}^{1}e^{-\frac{L^2}{8\beta^2}t^2}dt\,. \tag{9}$$

Table 1. Density Distributions

Distribution in transverse direction	Distribution in longitudinal direction			
	uniform	waterbag	parabolic	Gaussian
uniform	$\rho_{u,u}$	$\rho_{u,w}$	$\rho_{u,p}$	$\rho_{u,g}$
waterbag	$\rho_{w,u}$	$\rho_{w,w}$	$\rho_{w,p}$	$\rho_{w,g}$
parabolic	$\rho_{p,u}$	$\rho_{p,w}$	$\rho_{p,p}$	$\rho_{p,g}$
Gaussian	$\rho_{g,u}$	$\rho_{g,w}$	$\rho_{g,p}$	$\rho_{g,g}$

Substituting the density distributions of Eq. (8) into Eqs. (3) to (6), we can get the potentials induced by the cylinder of space charge with different density distributions. (For details of the derivation see Refs. [10] to [12]).

3 NONLINEAR FIELD ENERGY OF A CYLINDER OF SPACE CHARGE IN LINAC

The nonlinear field energy of the cylinder model of space charge in a waveguide of linac can be found by integrating φdq over the entire volume occupied by the space charge: [9]

$$W = 2\pi\int_{r=0}^{r=b}\int_{z=-\frac{L}{2}}^{z=\frac{L}{2}}\rho\varphi rdrdz\,. \tag{10}$$

Substituting the different density distributions of Eq.(8) and the potentials φ induced by these space charge in the regime ($|z|<L/2$) from Eq.(6) into Eq.(10), we get the self-field energy of the cylinder of space charge in waveguide as the following.

For a bunched beam with longitudinal uniform distribution and different distributions in transverse direction, we get:

$$W_{u,u} = \frac{2N^2q^2a^2}{\pi\varepsilon_0 Lb^2}\sum_{l=1}^{\infty}\frac{J_1^2(k_l b)}{(k_l a)^4 J_1^2(k_l a)}A_{kv} , \tag{11}$$

$$W_{w,u} = \frac{32N^2q^2a^4}{\pi\varepsilon_0 Lb^4}\sum_{l=1}^{\infty}\frac{J_2^2(k_l b)}{(k_l a)^6 J_1^2(k_l a)}A_{kv} , \tag{12}$$

$$W_{p,u} = \frac{1152N^2q^2a^6}{\pi\varepsilon_0 Lb^6}\sum_{l=1}^{\infty}\frac{J_3^2(k_l b)}{(k_l a)^8 J_1^2(k_l a)}A_{kv} , \tag{13}$$

$$W_{g,u} = \frac{N^2q^2}{2\pi\varepsilon_0 L}\sum_{l=1}^{\infty}\frac{e^{-k_l^2\alpha^2}}{(k_l a)^2 J_1^2(k_l a)}A_{kv} , \tag{14}$$

where

$$A_{kv} = 1-\frac{2}{k_l L}sh\frac{k_l L}{2}e^{-\frac{k_l L}{2}} . \tag{15}$$

For a bunched beam with longitudinal waterbag distribution and different distributions in transverse direction, we get:

$$W_{u,w} = \frac{18N^2q^2a^2}{\pi\varepsilon_0 Lb^2}\sum_{l=1}^{\infty}\frac{J_1^2(k_l b)}{(k_l a)^4 J_1^2(k_l a)}A_{wb} , \tag{16}$$

$$W_{w,w} = \frac{288N^2q^2a^4}{\pi\varepsilon_0 Lb^4}\sum_{l=1}^{\infty}\frac{J_2^2(k_l b)}{(k_l a)^6 J_1^2(k_l a)}A_{wb} , \tag{17}$$

$$W_{p,w} = \frac{10368N^2q^2a^6}{\pi\varepsilon_0 Lb^6}\sum_{l=1}^{\infty}\frac{J_3^2(k_l b)}{(k_l a)^8 J_1^2(k_l a)}A_{wb} , \tag{18}$$

$$W_{g,w} = \frac{9N^2q^2}{2\pi\varepsilon_0 L}\sum_{l=1}^{\infty}\frac{e^{-k_l^2\alpha^2}}{(k_l a)^2 J_1^2(k_l a)}A_{wb} , \tag{19}$$

where

$$A_{wb} = \frac{2}{15}-\frac{1}{3(k_l L/2)^2}+\frac{1}{(k_l L/2)^3}\left[1+\frac{1}{(k_l L/2)}\right]- -\frac{1}{(k_l L/2)^3}\left[1+\frac{1}{(k_l L/2)}\right]^2 e^{-\frac{k_l L}{2}}sh\frac{k_l L}{2} . \tag{20}$$

For a bunched beam with longitudinal parabolic distribution and different distributions in transverse direction, we get:

$$W_{u,p} = \frac{20N^2q^2a^2}{7\pi\varepsilon_0 Lb^2}\sum_{l=1}^{\infty}\frac{J_1^2(k_l b)}{(k_l a)^4 J_1^2(k_l a)}A_{pa} , \tag{21}$$

$$W_{w,p} = \frac{320N^2q^2a^4}{7\pi\varepsilon_0 Lb^4}\sum_{l=1}^{\infty}\frac{J_2^2(k_l b)}{(k_l a)^6 J_1^2(k_l a)}A_{pa} , \tag{22}$$

$$W_{p,p} = \frac{11520N^2q^2a^6}{7\pi\varepsilon_0 Lb^6}\sum_{l=1}^{\infty}\frac{J_3^2(k_l b)}{(k_l a)^8 J_1^2(k_l a)}A_{pa} , \tag{23}$$

$$W_{g,p} = \frac{5N^2q^2}{7\pi\varepsilon_0 L}\sum_{l=1}^{\infty}\frac{e^{-k_l^2\alpha^2}}{(k_l a)^2 J_1^2(k_l a)}A_{pa} , \tag{24}$$

where

$$A_{pa} = 1-\frac{3}{(k_l L/2)^2}+\frac{63}{2(k_l L/2)^4}+\frac{945}{2(k_l L/2)^6}+\frac{2835}{2(k_l L/2)^7}+\frac{2835}{2(k_l L/2)^8} -\frac{315}{2(k_l L/2)^5}\left[1+\frac{3}{(k_l L/2)}+\frac{3}{(k_l L/2)^2}\right]^2 e^{-\frac{k_l L}{2}}sh\frac{k_l L}{2} . \tag{25}$$

For a bunched beam with longitudinal Gaussian distribution and different distributions in transverse direction, we get:

$$W_{u,g} = \frac{2N^2q^2a^2}{\pi\varepsilon_0 Lb^2}\sum_{l=1}^{\infty}\frac{J_1^2(k_l b)}{(k_l a)^4 J_1^2(k_l a)}A_{ga} , \tag{26}$$

$$W_{w,g} = \frac{32N^2q^2a^4}{\pi\varepsilon_0 Lb^4}\sum_{l=1}^{\infty}\frac{J_2^2(k_l b)}{(k_l a)^6 J_1^2(k_l a)}A_{ga} , \tag{27}$$

$$W_{p,g} = \frac{1152N^2q^2a^6}{\pi\varepsilon_0 Lb^6}\sum_{l=1}^{\infty}\frac{J_3^2(k_l b)}{(k_l a)^8 J_1^2(k_l a)}A_{ga} , \tag{28}$$

$$W_{g,g} = \frac{N^2q^2}{2\pi\varepsilon_0 L}\sum_{l=1}^{\infty}\frac{e^{-k_l^2\alpha^2}}{(k_l a)^2 J_1^2(k_l a)}A_{ga} , \tag{29}$$

where

$$A_{ga} = \frac{k_l L}{8D^2}\int_{-1}^{1}\left[\left(1-|t|\right)\int_{-1}^{1}e^{-\frac{L^2}{8\beta^2}\left(\frac{1-|t|}{2}v+\frac{1+|t|}{2}\right)^2-\frac{k_l L}{2}\left(\frac{1-|t|}{2}v+\frac{1+|t|}{2}\right)}dv\right]\times \times e^{-\frac{L^2}{8\beta^2}t^2}ch\frac{k_l L}{2}tdt+\frac{k_l L}{8D^2}\int_{-1}^{1}e^{-\frac{L^2}{8\beta^2}t^2-\frac{k_l L}{2}|t|}\times \times\left[|t|\int_{-1}^{1}e^{-\frac{L^2}{8\beta^2}\left(\frac{|t|}{2}v+\frac{|t|}{2}\right)^2}ch\frac{k_l L}{2}\left(\frac{|t|}{2}v+\frac{|t|}{2}\right)dv\right]dt . \tag{30}$$

4 NUMERICAL RESULTS OF NONLINEAR FILED ENERGY

As an example, take a cylinder model of space charge with total charge $Nq=6\times10^{-9}$ Coulomb in a surrounding cylinder with radius a=0.015m. The calculations of nonlinear field energy W versus b/a of the bunch and wall radius and b/L of the bunch radius and length are carried out. The related plots for different transverse distributions combining different longitudinal distributions are shown in Figures 2 and 3. As can be seen, first, the all nonuniform particle distributions have more nonlinear field energy than that of the equivalent uniform beam. Second, the nonlinear field energies increase as b/L increases and decrease as b/a increases for all distributions. Thirdly, from Fig.2, it can be seen that the smaller the b/a, the bigger the nonlinear field energy W. Therefore, it can be predicted, the nonlinear field energy tends its maximum in free space for a space charge bunch with any distribution.

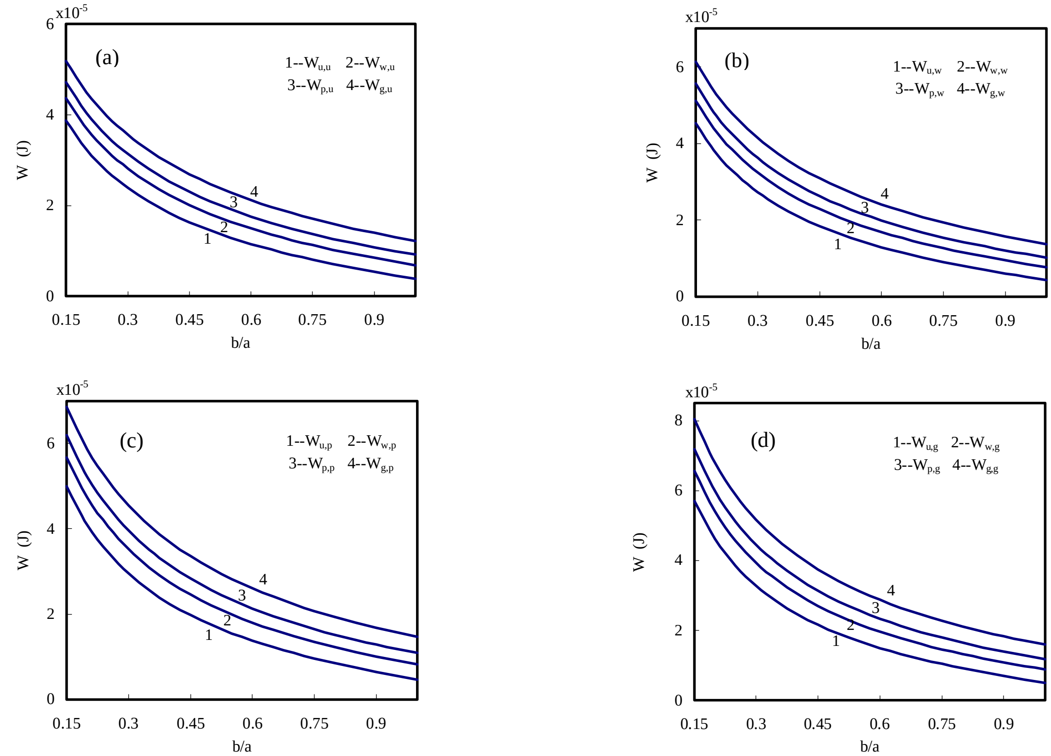

Figure 2. Field energy W versus b/a of the bunch and wall radius for different transverse distributions and (a) longitudinal uniform distribution; (b) longitudinal waterbag distribution; (c) longitudinal parabolic distribution; (d) longitudinal Gaussian distribution.

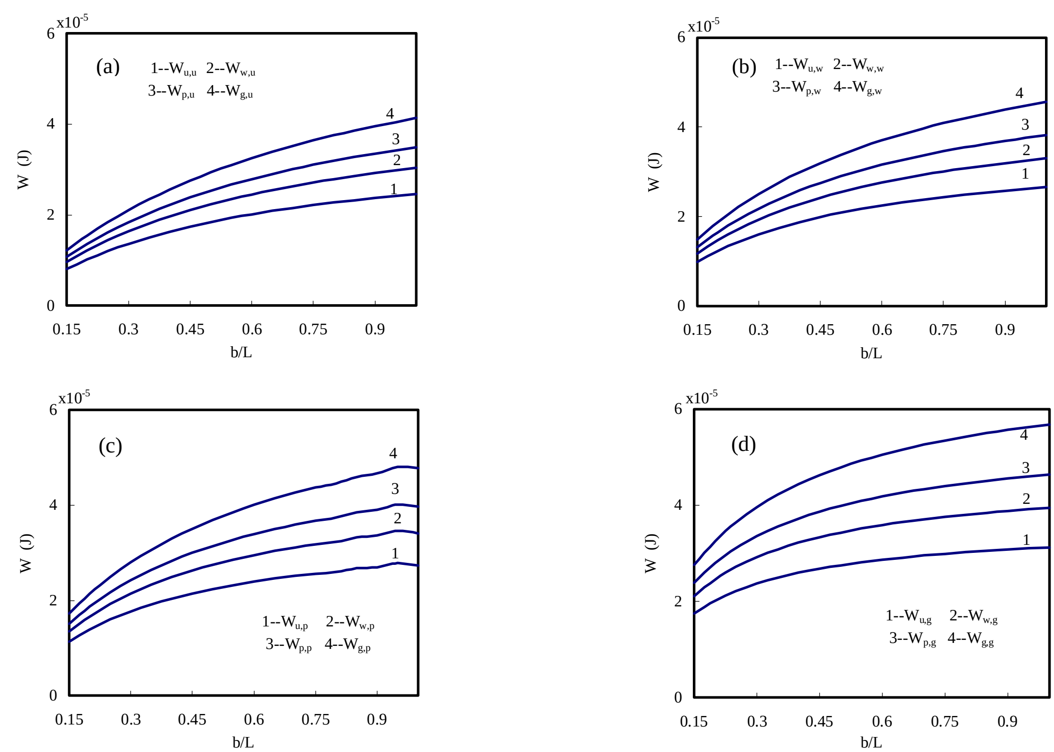

Figure 3. Field energy W versus b/L of the bunch radius and length for different transverse distributions and (a) longitudinal uniform distribution; (b) longitudinal waterbag distribution; (c) longitudinal parabolic distribution; (d) longitudinal Gaussian distribution.

5 EMITTANCE GROWTH FOR A BUNCHED BEAM IN LINAC

The generalized three-dimensional equation for the emittance and field energy of high-current beams in periodic focusing structure was derived by Hofmann and Struckmeier [5], and can be written for three degrees of freedom *x*, *y* and *z* (with linear focusing in each plane) as:

$$\frac{1}{x^2}\frac{d}{ds}\varepsilon_x^2+\frac{1}{y^2}\frac{d}{ds}\varepsilon_y^2+\frac{1}{z^2}\frac{d}{ds}\varepsilon_z^2=-\frac{32}{m\gamma^3 v^2 N}\frac{d}{ds}\left(W-W_u\right), \qquad (31)$$

where $s=v\cdot t$, ε_x , ε_y and ε_z present rms emittance in *x*, *y* and *z* directions, respectively, W and W_u are the space charge field energies of the real beam and equivalent uniform beam. The equation allows one to estimate the total emittance growth if the change of nonlinear field energy can be predicted.

In linear accelerator, the real beams have to be considered in some channels. And hence, the nonlinear field energy of the beam should be calculated in a surrounding structure. It can be seen clearly, from the Fig.2 and Fig.3, the nonlinear field energies are not only dependent of the ratio b/L of the bunch radius and length, but also dependent of the ratio b/a of the bunch and wall radius. It is also true for the free energy. Letting the quantity U_n:

$$U_n=\left(W-W_u\right)/W_u \ . \qquad (32)$$

U_n is a measure of the nonuniformity of the charge density and represents the field energy which can be converted into particle kinetic energy as the distribution tends to become more homogeneous. And hence it leads the emittance growth. As some examples, taking $W_u = W_{u,u}$, and W equals to $W_{w,w}$, $W_{p,w}$, $W_{g,w}$, $W_{w,p}$, $W_{p,p}$, and $W_{g,p}$ respectively, we obtain U_n versus b/a for these distributions as shown in Fig.4.

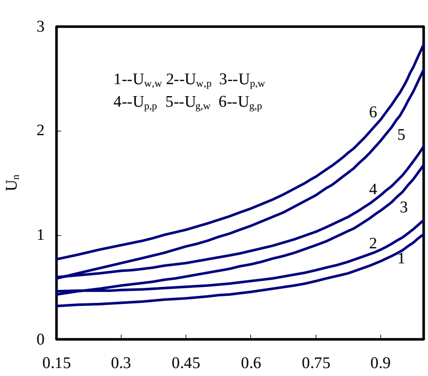

Figure 4. Free energy U_n versus b/a for some distributions.

Obviously, the emittance growth caused by the free energy U_n will be also dependent of the ratio of the bunch and wall radius, and may tend its minimum in free space. Therefore, it may be not enough to form a true estimation of the emittance growth if the change of nonlinear field energy is predicted with the value in free space in stead of that in a channel of linac.

REFERENCES

[1] M. Lapostolle, Possible emittance increase through filamentantion due to space charge in continuous beams, IEEE Trans. Nucl. Sci. NS-18, 1101 (1971).

[2] F. Sacherer, RMS envelope equations with space charge, IEEE Trans. Nucl. Sci. NS-18, 1105 (1971).

[3] J. Struckmeier, J. Klabunde and M. Reiser, On the stability and emittance growth of different particle phase space distributions in a long magnetic quadrupole channel, Part. Acc. 15, 47 (1984).

[4] T. P. Wangler, K. R. Crandall, R. S. Mills, and M. Reiser, Relationship between field and rms emittance in intense particle beams, IEEE Trans. Nucl. Sci. NS-32, 2196(1985).

[5] I. Hofmann and J. Struckmeier, Generalized three-dimensional equations for the emittance and field energy of high-current beams in periodic focusing structures, Part. Acc. 21, 69 (1987).

[6] T. P. Wangler, P. Lapostolle and A. Lombardi, Space-charge-induced emittance growth in an elliptical beam with a parabolic density distribution, Conference record of the IEEE 1993 particle Accelerator Conference, 93CH3279-7, p.3606.

[7] M. Reiser, Theory and Design of Charged Particle Beams, John Wiley & Sons, Inc., New York, 1994.

[8] Chen Yinbao, Xie Xi, Space charge effects of nonuniform density distribution in waveguide, Chinese Journal of Nuclear Physics, 1979, 1:107-128

[9] J.Hechtel, The effect of potential beam energy on the performance of linear beam devices", IEEE Trans. on electron devices, 1970, 15:999-1009

[10] Chen Yinbao, Nonlinear space charge effects of bunched beam in linac, China Nuclear Science & Technology Report, CNIC-00592, IAE-0100, 1992.

[11] Chen Yinbao, Fu Sinian, Huang Zhibin at el., Space charge effects of bunched beam with nonuniform distributions in both longitudinal and transverse directions, Chin. J. Nucl. Phys. 1995, 17(1):80.

[12] Chen Yinbao, Fu Sinian, Huang Zhibin at el., Space charge effects of bunched beam with Gaussian distribution in linac, Chin. J. Nucl. Phys. 1995, 17(2):174.