ANALYTICAL TREATMENT OF SINGLE BUNCH STABILITY IN A LINAC

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Abstract

Single bunch stability is analysed by solving the equation of motion of the particles travelling in a linac, for a Gaussian distribution of charge, a linear variation of the transverse wakefield along the bunch, a smooth focusing and negligible acceleration. The treatment is based on a non-standard perturbation expansion that has been specifically developed for this study and preserves at each order the intrinsic detuning likely to stabilise the resonant beam break-up. It provides a closed expression for the tune shift along the bunch resulting from BNS damping and autophasing, methods proposed in the past to control the emittance, and a first-order solution for the transverse off-sets within the bunch. The analytic result obtained makes it possible to study the behaviour of the solution and compute the emittance dilution in specific cases. The present theory is a useful complement to the numerical simulations done with the MUSTAFA code in the Compact Linear Collider scheme (CLIC). It also gives an interesting as well as comprehensive view of the physics involved in the single-bunch motion and the damping of the instability.

1 EQUATION OF MOTION

Since in most linear colliders a flat beam design (low vertical to horizontal beam size ratio) is used, emittance blow up due to transverse wakefields is most critical in the vertical plane. Disregarding acceleration, using a weak focusing model for the betatron motion and assuming a linearly varying wakefield within a single bunch the equation of motion reads as (refs [1, 2])

$$\frac{\partial^2 x(s,z)}{\partial s^2} + q^2[1+\Delta k(z)]x(s,z) = \frac{CW_0}{\gamma_0 l_B}\int_0^z \rho(z^*)(z-z^*)x(s,z^*)dz^* \qquad (1)$$

The two independent variables s and z describe the position of the bunch inside the linac, and the position inside the bunch, respectively. The quantity $q = 1/\bar{\beta}_y$ is the weak focusing tune, $\Delta k(z)$ a z-dependent additional focusing force (arising from a correlated energy spread and/or RF quadrupoles) and $\rho(z)$ the line charge density distribution. The constant C is defined by $C = 4\pi\epsilon_0 r_e N$ where N is the number of particles inside the bunch and r_e is the classical electron radius. W_0 is the value of the transverse wake at the tail of a truncated bunch in units of $V/(Asm^2)$. For $\rho(z)$ a truncated Gaussian distribution ($\pm 2\sigma_z$) has been used. In order to facilitate the analysis, the Gaussian has been replaced by its 4-th order Chebyshev approximation within $\pm 2\sigma_z$ which results into an error of not more than 4%. After normalisation the thus approximate expression for ρ becomes

$$\rho(z) = \frac{75}{46 l_B}\left[1 - \frac{41}{100}\left(\frac{4z}{l_B}-2\right)^2 + \frac{1}{20}\left(\frac{4z}{l_B}-2\right)^4\right] \qquad (2)$$

where $l_B = 4\sigma_z$.

We choose to deal with the effect of an initial offset α_0 as well as an initial slope along z. Although we do not consider here randomly misaligned quadrupoles and accelerating cavities, it has to be noted that these off-set and slope represent well the misalignment of a single component of the linac. The initial conditions are then

$$x(0,z) = \alpha_0 + \alpha_1 z \qquad (3)$$

$$\frac{dx}{ds}(0,z) = 0 \qquad (4)$$

2 SEPARATION OF VARIABLES AND AUTOPHASING

Equation (1) is a linear, partial, homogeneous integro-differential equation of second order. This type of equation can often be solved analytically by separating the two independent variables, i.e. s and z. Rewriting $x(s,z)$ as $x = X(s) + y(s,z)$ and performing some algebra leads to the following new equations for X and y

$$\frac{d^2X}{ds^2} + q^2 X = 0 \qquad (5)$$

$$\frac{\partial^2 y}{\partial s^2} + q^2[1+\Delta k(z)]y = X(s)\left[-q^2\Delta k(z) + \frac{CW_0}{\gamma_0 l_B}\int_0^z \rho(z^*)(z-z^*)dz^*\right] + \frac{CW_0}{\gamma_0 l_B}\int_0^z \rho(z^*)(z-z^*)y(s,z^*)dz^* \qquad (6)$$

The coherent motion (5) is given by the unperturbed betatron equation and its solution according to the initial conditions given above is

$$X(s) = \alpha_0 \cos qs = \alpha_0 \cos\frac{s}{\bar{\beta}_y} \qquad (7)$$

Considering the case of no z dependent focusing across the bunch ($\Delta k(z) = 0$), we face a resonant situation due to the fact that the frequency q of the unperturbed betatron motion appears on the right hand side and generates a secular solution in s. This is related to the well-known head to tail instability of a single bunch traveling through a structure with wakefields. In order to suppress the resonance excitation term, it is necessary to introduce a tune spread along the bunch [3] cancelling the coefficient of $X(s)$ in Eq. (6).

$$\Delta k(z)_{AUTO} = \frac{CW_0}{\gamma_0 l_B q^2}\int_0^z \rho(z^*)(z-z^*)dz^* \qquad (8)$$

In this paper we do not specify the mechanism creating the detuning (RF quadrupoles or correlated energy spread). However, in order to also study the bunch dynamics in the case of no correction or only partial correction through a z-dependent focusing, the actual detuning is defined as

$$\Delta k(z) = \lambda \Delta k(z)_{AUTO} = \lambda\frac{CW_0}{\gamma_0 l_B q^2}\int_0^z \rho(z^*)(z-z^*)dz^* \qquad (9)$$

where $\lambda = 0$ means no correction while $\lambda = 1$ corresponds to the **autophasing condition** (8) (resonance suppressed). Inserting our definition (9) into Eq. (6) gives

$$\frac{\partial^2 y}{\partial s^2} + q^2\left[1 + \frac{\lambda CW_0}{\gamma_0 l_B q^2}\int_0^z \rho(z^*)(z-z^*)dz^*\right] y =$$
$$(\lambda - 1)\frac{CW_0}{\gamma_0 l_B}X(s)\int_0^z \rho(z^*)(z-z^*)dz^* +$$
$$\frac{CW_0}{\gamma_0 l_B}\int_0^z \rho(z^*)(z-z^*)y(s,z^*)dz^* \qquad (10)$$

3 PERTURBATIVE SOLUTION

Introducing the normalised bunch coordinate $\zeta = z/l_B$ and further separating y such as $y(s,\zeta) = \alpha_1 l_B \zeta \cos qs + v(s,\zeta)$ results in

$$\frac{\partial^2 v}{\partial s^2} + \bar{q}^2(\zeta)v = \frac{CW_0}{\gamma_0 l_B}\cos qs \times$$
$$\left[[\alpha_0(1-\lambda) - \alpha_1 l_B \zeta\lambda]\int_0^\zeta l_B^2 \rho(l_B\zeta)(\zeta-\zeta^*)d\zeta^*\right.$$
$$\left. + \alpha_1\int_0^\zeta l_B^3\zeta^*\rho(l_B\zeta)(\zeta-\zeta^*)d\zeta^*\right]$$
$$+ \frac{CW_0\epsilon}{\gamma_0 l_B}\int_0^\zeta l_B^2\rho(l_B\zeta^*)(\zeta-\zeta^*)v(s,\zeta^*)d\zeta^* \qquad (11)$$

where

$$\bar{q}^2(\zeta) = q^2\left[1 + \lambda E\frac{CW_0}{\gamma_0 l_B q^2}\int_0^\zeta l_B^2\rho(l_B\zeta^*)(\zeta-\zeta^*)d\zeta^*\right] \qquad (12)$$

In order to avoid secular terms and preserve the detuning of the oscillatory motion we use a specific **partial perturbation expansion** of the solution $v(s,\zeta)$. It consists of separating the perturbation of the (analytically) solvable part of the equation of motion (11), marked with E in (12), from the integral driving-term on the right hand side of (11) marked with ϵ. Both ϵ and E indicate that the associated terms contain products of the wakefield W_0 with the oscillation amplitude due to the wakefields $v(s,\zeta)$ and are perturbations w.r.t. the rest of the equation. The expansion is only done w.r.t. ϵ and not w.r.t. E describing the z-dependent tune shift. Then the perturbation series reads as $v = v^{(0)}(s,\zeta;E) + \epsilon v^{(1)}(s,\zeta,E) + \cdots$ and no resonant terms arise at any order. At the end of the computation, both ϵ and E are set to unity. A detailed description and justification of the partial expansion method introduced for this study is given in [2]. Setting $\epsilon = 0$ in eq. (11) leads to a linear, inhomogeneous differential equation of second order and its solution is

$$v^{(0)} = \left[\alpha_0\frac{\lambda-1}{\lambda} + \left(\frac{3\zeta}{7\lambda} - \zeta\right)\alpha_1 l_B\right][\cos qs - \cos(\bar{q}(\zeta)s)] \qquad (13)$$

with

$$\bar{q}(\zeta) = q\left[1 + \frac{\lambda CW_0}{\gamma_0 q^2}\times\right. \qquad (14)$$
$$\left.\left(\frac{16}{23}\zeta^6 - \frac{48}{23}\zeta^5 + \frac{79}{46}\zeta^4 + \frac{1}{23}\zeta^3 + \frac{3}{23}\zeta^2\right)\right]^{\frac{1}{2}}$$

The equation for the first order perturbation contribution $v^{(1)}$ becomes

$$\frac{\partial^2 v^{(1)}}{\partial s^2} + \bar{q}^2(\zeta)v^{(1)} =$$
$$\frac{CW_0 l_B}{\gamma_0}\int_0^\zeta \rho(l_B\zeta)(\zeta-\zeta^*)v^{(0)}(\zeta^*,s)d\zeta^* \qquad (15)$$

The detailed solution of this equation is given in Ref. [2].

As a first example, Fig. 1 shows a typical solution (to order zero of the perturbation) of the form $y = \alpha_1\zeta l_B\cos qs + v^0(s,\zeta)$ at a distance of 520 m downstream of the linac. The increasing frequency of the incoherent bunch oscillations from the head ($\zeta = 0$) to the tail ($\zeta = 1$) of the bunch becomes clearly visible.

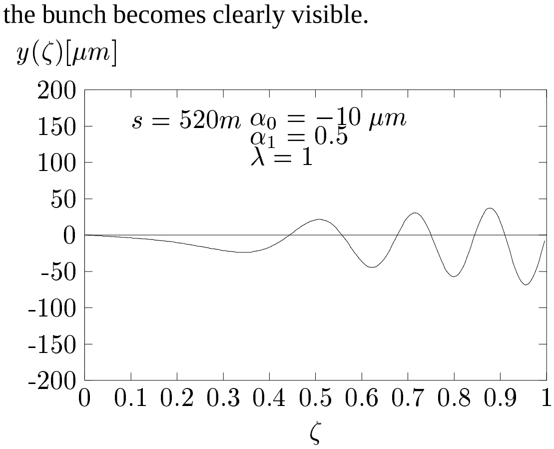

Figure 1: Autophasing solution in CLIC

Next we show in Fig. 2 the solution (including the first order term $v^{(1)}$ of the perturbation) for $\lambda = 0$, i.e. in the absence of detuning along the bunch. While the full line represents the analytical solution, the points indicate the results obtained with the tracking code MUSTAFA [4].

By comparison with the detuned example of Fig. 1, the amplitude of the oscillation increases significantly because of the resonant effect. However, some residual detuning remains visible due to the influence at large amplitudes of the last integral term in Eq. (11).

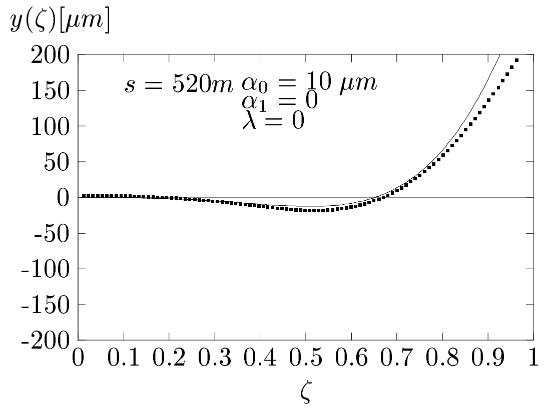

Figure 2: Resonant solution in CLIC at $s = 520$ m

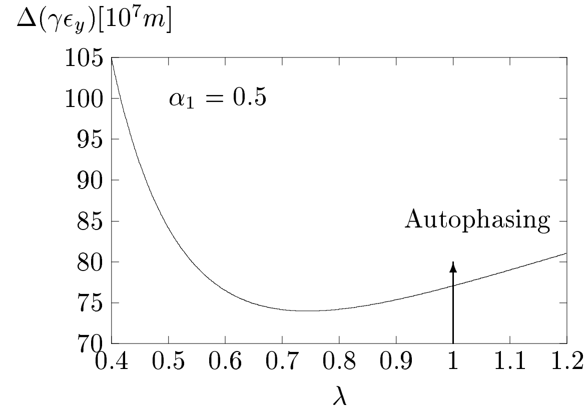

Figure 3: Asymptotic emittance growth as function of λ

4 EMITTANCE DILUTION

Since a low emittance beam is needed at the interaction point to provide high luminosity collisions, it is interesting to study the emittance dilution due to wakefields in the accelerating structures of the collider. If we consider the emittance increase due to transverse wakes in a single bunch, the total normalised emittance at the end of the main linac is given by

$$\gamma\epsilon_{tot} = \gamma\epsilon_{inj} + \Delta(\gamma\epsilon_y) \tag{16}$$

where

$$\Delta(\gamma\epsilon_y) = l_B\gamma_0 \int_0^1 \rho(\zeta) \left[q y^2(s,\zeta) + \frac{1}{q}\left(\frac{\partial y}{\partial s}(s,\zeta)\right)^2 \right] d\zeta \tag{17}$$

and $y = \alpha_1 l_B \zeta \cos qs + v(s,\zeta)$. Instead of v we use $v^{(0)}(s,\zeta)$ as given in Eq. (13) since it is believed to give the strongest contribution. Although y has then a relatively simple form, the integral in (17) becomes non elementary, leading to complicated expressions of trigonometric and Fresnel functions. However, it can be demonstrated that the emittance in the case of an initial offset tends to an asymptotic value as s goes to infinity. It is straightforward to compute this limit by only considering slowly oscillating terms in ζ as s increases and averaging the fast oscillating terms before performing the quadrature. As above, all the details are described in Ref. [2]. The result for the asymptotic emittance becomes

$$\lim_{s\to\infty}(\gamma_0\epsilon_y) = \gamma_0\epsilon_{inj} + \frac{a_{-2}}{\lambda^2} + \frac{a_{-1}}{\lambda} + a_0 + a_1\lambda + a_2\lambda^2 \tag{18}$$

where the parameters a_{-2} to a_2 are polynomial expressions of the transverse wakefield W_0 with coefficients that are rational functions of α_0 , α_1 , q and l_B. They are all listed in Ref. [2]. The form of the expression (18) makes it obvious that ϵ_y must have a certain minimum as a function of λ, since the first and second terms decrease with λ while the last two increase. Evidently $\lambda = 0$ leads to an infinite asymptotic emittance due to the resonance effect.

Fig. 3 gives an illustration of this effect for the case of $\alpha_0 = -10\ \mu m$, $\alpha_1 = 0.5$ and $0.4 < \lambda < 1.2$.

The same figure 3 clearly shows for $\alpha_1 = 0.5$ that **the autophasing condition is not leading in general to the minimum single bunch emittance**. Instead, the minimum is shifted to a lower value of λ where two effects are best balanced: namely an increase of the decoherence of the bunch with λ that leads to an emittance growth and a rise of the distance from the resonance with λ that induces an emittance reduction. The actual (flat) minimum of the emittance in this example appears at a value of λ near 75% of the one corresponding to the autophasing condition.

5 CONCLUSIONS

The equation of the transverse single-bunch motion has been analytically solved for initial off-set and slope along the bunch, similar to those generated by the misalignment of a single linac element. Weak focusing is used and the acceleration within a linac sector is not included. The transverse wakefield along the bunch is taken as linear and the charge density approximated by Chebyshev polynomials. The transverse displacement $x(s)$ is split into a coherent term $X(s)$ and a part $y(s,z)$ that depends on the position z in the bunch. A particular partial perturbation treatment worked out by the authors is then applied in order to keep the detuning property through all orders and prevent the creation of artificial resonances. In this way, zero and first order solutions for x as well as the asymptotic emittance dilution for $s \to \infty$ have been derived as functions of the fraction λ of the wakefield that is damped. They all agree with the results of the code MUSTAFA and confirm the existence of a given λ where the single-bunch emittance blow-up is minimum. This minimum is proven not to correspond generally with autophasing.

6 REFERENCES

[1] See for instance A.W.Chao, B.Richter, C.Y.Yao, *Nucl. Instr. Meth,*, **178**, 1 (1980), and H.Henke, W.Schnell, report CERN-LEP-RF 86-18 (1986).

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