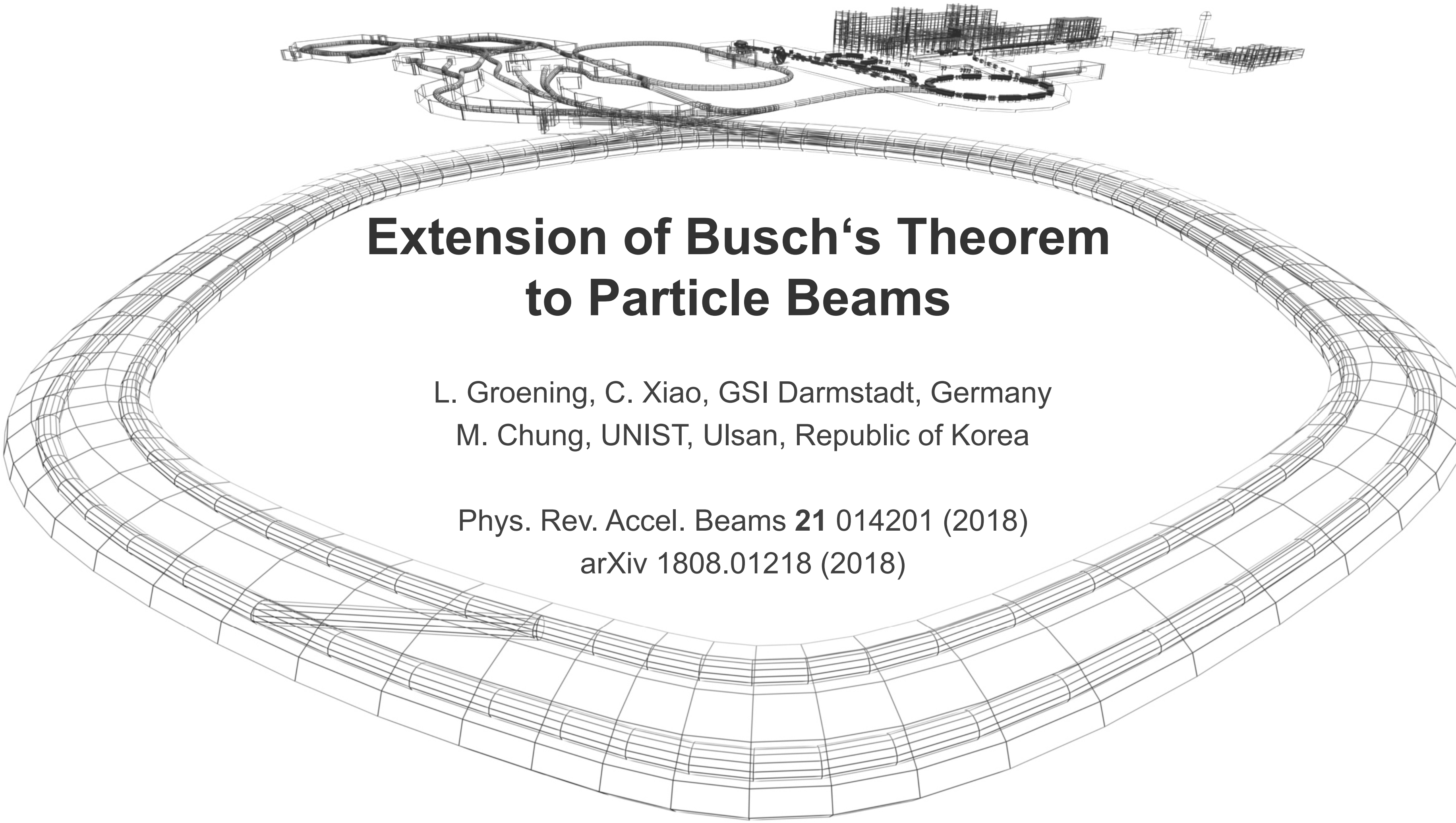

Extension of Busch‘s Theorem to Particle Beams

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Phys. Rev. Accel. Beams **21** 014201 (2018)
arXiv 1808.01218 (2018)

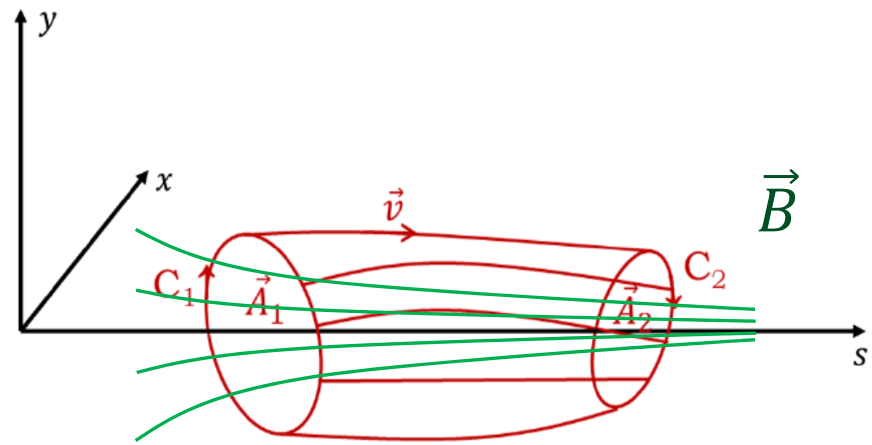
Outline

- **Busch theorem for single particle**
- **Eigen emittances**
- **Busch theorem extended to accelerated beams**
- **Application to emittance shaping of:**
    - **electron beams**
    - **ion beams**

Busch theorem for single particle (1926)

$$m\gamma r^2\dot{\theta} + \frac{eq}{2\pi}\psi = \text{const}$$

orbital angular momentum +
magn. flux through cyclotron motion area = const



Theorem states preservation of axial canonical angular momentum in presence of axial B-field. “Canonical“ means that the magnetic vector potential is included into the momenta (conjugated momenta)

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)},$$

$$p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)},$$

generalized Busch theorem

$$\frac{eq}{m\gamma}\psi + \int_A [\vec{\nabla} \times \vec{v}]d\vec{A} = const$$

Beam eigen emittances (1992)

Beam matrix of coupled beam:

$$C = \begin{bmatrix} \langle xx\rangle & \langle xx'\rangle & \langle xy\rangle & \langle xy'\rangle \\ \langle x'x\rangle & \langle x'x'\rangle & \langle x'y\rangle & \langle x'y'\rangle \\ \langle yx\rangle & \langle yx'\rangle & \langle yy\rangle & \langle yy'\rangle \\ \langle y'x\rangle & \langle y'x'\rangle & \langle y'y\rangle & \langle y'y'\rangle \end{bmatrix}$$

$$\varepsilon_1 = \frac{1}{2}\sqrt{-tr[(CJ)^2] + \sqrt{tr^2[(CJ)^2] - 16det(C)}}$$

$$\varepsilon_2 = \frac{1}{2}\sqrt{-tr[(CJ)^2] - \sqrt{tr^2[(CJ)^2] - 16det(C)}}$$

eigen emittances are the two transverse rms emittances, which the beam acquires after hor/ver coupling has been removed

$$J := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

preserved by linear Hamiltonean (symplectic) beam line elements as:
drift, quadrupole, dipole, solenoid, rf-gaps (and tilts of them)

Preservation of eigen emittances

Busch theorem: preservation of axial angular momentum using conjugate momenta

Extended theorem: preservation of eigen emittances using conjugate momenta

$$\tilde{C} = \begin{bmatrix} \langle x^2\rangle & \langle xp_x\rangle & \langle xy\rangle & \langle xp_y\rangle \\ \langle xp_x\rangle & \langle p_x^2\rangle & \langle yp_x\rangle & \langle p_xp_y\rangle \\ \langle xy\rangle & \langle yp_x\rangle & \langle y^2\rangle & \langle yp_y\rangle \\ \langle xp_y\rangle & \langle p_xp_y\rangle & \langle yp_y\rangle & \langle p_y^2\rangle \end{bmatrix}$$

$$p_x := x' + \frac{A_x}{(B\rho)} = x' - \frac{yB_s}{2(B\rho)}$$
$$p_y := y' + \frac{A_y}{(B\rho)} = y' + \frac{xB_s}{2(B\rho)}$$

$\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2 = \text{const.}$ and $A := \sqrt{\langle x^2\rangle\langle y^2\rangle - \langle xy\rangle^2}$ gives:

$$(\varepsilon_1 - \varepsilon_2)^2 + \left[\frac{AB_s}{(B\rho)}\right]^2 + 2\frac{B_s}{(B\rho)}\underbrace{[\langle y^2\rangle\langle xy'\rangle - \langle x^2\rangle\langle yx'\rangle + \langle xy\rangle(\langle xx'\rangle - \langle yy'\rangle)]}_{=\,0 \text{ for uncoupled beam}} = \text{const},$$

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= 0 for uncoupled beam

being equivalent to

$$(\varepsilon_1 - \varepsilon_2)^2 + \frac{\psi}{(B\rho)^2 eq}\int_A \left[\vec{\nabla} \times (\vec{\mathcal{P}} + eq\vec{A})\right] d\vec{A} = const$$

extended Busch theorem

diff. of eigen emitt. + canonical vorticity flux = const

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original Busch theorem

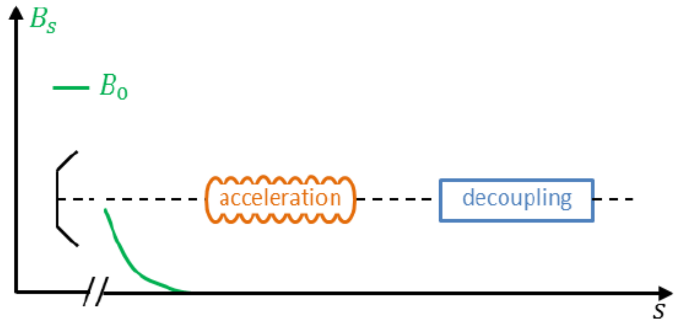
$$\frac{eq}{m\gamma}\psi + \int_A [\vec{\nabla} \times \vec{v}] d\vec{A} = const$$

Flat beam experiment @ FERMILAB

test accelerator at FERMILAB demonstrated $\varepsilon_y / \varepsilon_x = 100$:

- create beam at photo cathode being immersed into $B_s = B_0$
- reduce B_s to zero, accelerate, and decouple x/y - planes

P. Piot et al., Phys. Rev. ST Accel. Beams **9** 031001 (2006)



at cathode beam is symmetric:

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- eigen (rms) emittances are equal: $\varepsilon_{1/x} = \varepsilon_{2/y}$
- immersed into B-flux
- no x/y coupling → $\frac{2B_s}{(B\rho)}[\ldots] = 0$

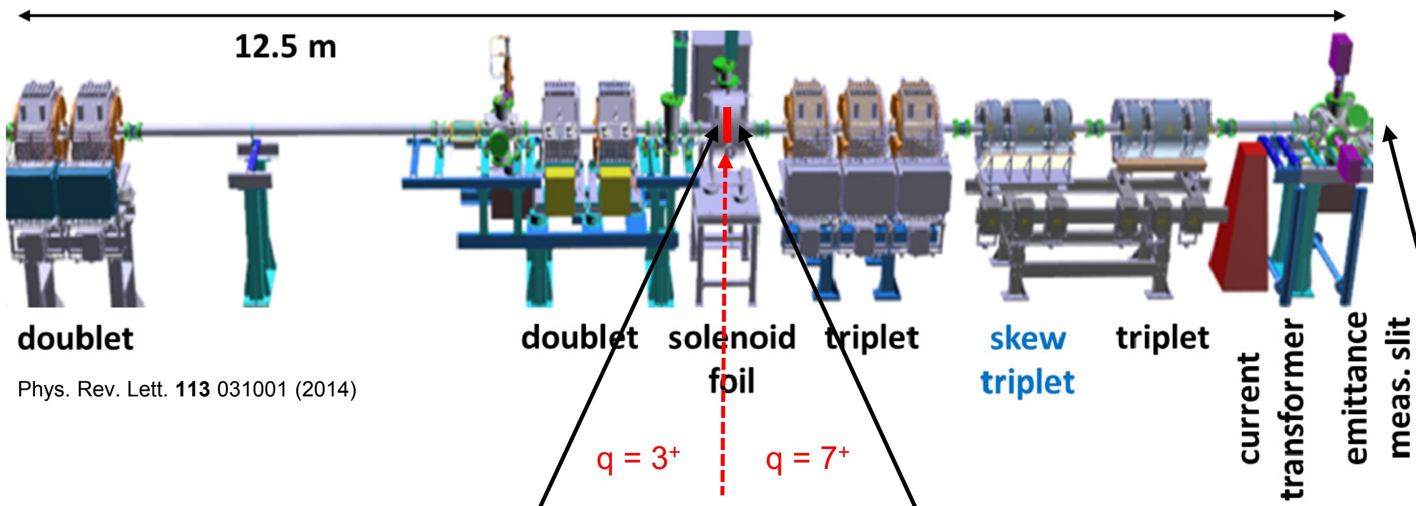
final beam:

- eigen = rms emittances: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- eigen (rms) emittances differ: $\varepsilon_{1/x} \neq \varepsilon_{2/y}$
- no B-flux
- no x/y coupling → $\frac{2B_s}{(B\rho)}[\ldots] = 0$

$$0 + \left[\frac{eB_0A_0}{mc}\right]^2 + 0 = (\varepsilon_{nfx} - \varepsilon_{nfy})^2 + 0 + 0$$

$$L := (eB_0A_0)/(2m\gamma\beta c) \rightarrow \varepsilon_{nfx/y} = \pm L\beta\gamma + \sqrt{(L\beta\gamma)^2 + {\varepsilon_{4d}}^2}$$

Transv. emittance transfer exp. @ GSI



- eigen = rms emitt.: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- measured: $\varepsilon_{x,3+}$ and $\varepsilon_{y,3+}$
- no B-flux
- no x/y coupling → $\frac{2B_s}{(B\rho)}[\ldots] = 0$

- eigen ≠ rms emitt.: $\varepsilon_{1f/2f} \neq \varepsilon_{x/y}$
- eigen emittances differ: $\varepsilon_{1f} \neq \varepsilon_{2f}$
- B-flux
- x/y coupling → $\frac{2B_s}{(B\rho)}[\ldots] \neq 0$

- eigen = rms emitt.: $\varepsilon_{1/2} = \varepsilon_{x/y}$
- to be determined: $\varepsilon_{x,7+}$ and $\varepsilon_{y,7+}$
- no B-flux
- no x/y coupling → $\frac{2B_s}{(B\rho)}[\ldots] = 0$

$$(\varepsilon_{x,7+} - \varepsilon_{y,7+})^2 = (\varepsilon_{x,3+} - \varepsilon_{y,3+})^2 + (A_f B_0)^2 \left[\frac{1}{(B\rho)_{7+}} - \frac{1}{(B\rho)_{3+}}\right]^2$$

emittance shaping