

WAVE BREAKING AND PARTICLE JETS IN INHOMOGENEOUS BEAMS*

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Abstract

We analyze the dynamics of inhomogeneous, magnetically focused high-intensity beams of charged particles. While for homogeneous beams the whole system oscillates with a single frequency, any inhomogeneity leads to propagating transverse density waves which eventually result in a singular density build up, causing wave breaking and jet formation. The theory presented in this paper allows to analytically calculate the time at which the wave breaking takes place. It also gives a good estimate of the time necessary for the beam to relax into the final stationary state consisting of a cold core surrounded by a halo of highly energetic particles.

INTRODUCTION

It is well known that magnetically focused beams of charged particles can relax from non-stationary into stationary flows with the associated particle evaporation [1]. Gluckstern [2] showed that initial oscillations of mismatched beams induce formation of large scale resonant islands [3, 4] beyond the beam border: beam particles are captured by the resonant islands resulting in emittance growth and relaxation. A closely related question concerns the mechanism of beam relaxation and the associated emittance growth when the beam is not homogeneous. On general grounds of energy conservation one again concludes that beam relaxation takes place as the coherent fluctuations of beam inhomogeneities are converted into microscopic kinetic energy [1, 5]. However, unlike in the former case for which the specific resonant mechanism is well understood, for inhomogeneous systems a more detailed description of the processes involved must still be explored. This is the goal of the present paper.

We find that the relaxation comes about as a consequence of breaking of density waves followed by ejection of fast particle jets. Jets are formed by particles moving in-phase with the macroscopic density fluctuations. They draw their energy from the propagating wave fronts and convert it into microscopic kinetic energy. For strongly inhomogeneous beams, we find that it is the wave breaking and jet production which are the primary mechanisms responsible for the beam relaxation.

THE MODEL AND ANALYSIS

We consider solenoidal focusing of space-charge dominated beams propagating along the transport axis, defined as the z axis, of our reference frame. The beam is initially cold with vanishing emittance and is azimuthally symmetric around the z axis. Prior to the appearance of density singularities one uses a cold fluid description for which Lagrangian coordinates are appropriate. In these coordinates, the transverse radial position r of a beam element is governed by [6, 7, 8]

$$r'' = -\kappa r + \frac{Q(r_0)}{r}; \quad (1)$$

the prime indicates derivative with respect to the longitudinal z coordinate which for convenience we shall also refer to as “time”. The focusing factor is $\kappa \equiv (qB/2\gamma m\beta c^2)^2$, where B is the axial, constant, focusing magnetic field; $Q(r_0) = KN(r_0)/N_t$, is the measure of the charge contained between the origin at $r = 0$ and the initial position $r(z = 0) = r_0$, N_t is the total number of beam particles per unit axial length, $N(r_0)$ is the number of particles up to r_0 , and $K = N_t q^2 / \gamma^3 m \beta^2 c^2$ is the beam perveance. q and m denote the beam particle charge and mass, respectively; $\gamma = (1 - \beta^2)^{-1/2}$ is the relativistic factor where $\beta = v_z/c$ and v_z is the constant axial beam velocity and c is the speed of light. Note that r_0 is in fact the Lagrange coordinate of the fluid element [9] which means that as long as the fluid description remains valid, the amount of charge seen by the fluid element inside the region $0 < r \leq r(z)$ remains unaltered at $Q(r_0)$, independent of time z . This is of fundamental importance since from the Gauss law this is the charge that exerts the force on the fluid element. We will consider beams starting from a static initial condition, $r'(0) = 0$. The formal solution to the fluid dynamics Eq. (1) is $r = r(z, r_0)$. This can be calculated explicitly using the Lindstedt-Poincaré perturbation theory. As long as the fluid picture applies, for small amplitude fluctuations around the stable equilibrium $r_{eq}(r_0) = \sqrt{Q(r_0)/\kappa}$ we obtain

$$r(r_0, z) = r_{eq} \left\{ 1 + A/r_{eq} \cos(\omega z) + (1/3)(A/r_{eq})^2 \times [2 + \cos(\omega z)] \sin^2(\omega z/2) + \mathcal{O}[(A/r_{eq})^3] \right\}, \quad (2)$$

where $A(r_0) = r_0 - r_{eq}$ is the amplitude of oscillations, $\omega(r_0) = \omega_0 + \sqrt{\kappa} A^2 / (6\sqrt{2} r_{eq}^2)$ is the renormalized r_0 -dependent frequency, and $\omega_0 = \sqrt{2\kappa}$ is the unperturbed frequency. The time evolution of the beam

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density can be obtained as follows. For a beam of initial cross sectional density $n_0(r)$, the amount of charge δQ between two concentric circles of radii r_0 and $r_0 + \delta r_0$ (δr_0 small) is $\delta Q = 2\pi r_0 \delta r_0 (K/N_t) n_0(r_0)$. Since this charge is conserved, $\delta Q = 2\pi r \delta r (K/N_t) n(r) = 2\pi r (\partial r / \partial r_0) (K/N_t) n(r) \delta r_0$, and the transverse beam density at any future time z is therefore,

$$n(r) = n_0(r_0) (r_0/r) (\partial r / \partial r_0)^{-1} \Big|_{r_0=r_0(r,z)}. \quad (3)$$

For a given position r and an axial coordinate z , the initial position r_0 of a beam element can be uniquely determined as the inverse function $r_0 = r_0(r, z)$ of Eq. (2). This ceases to be the case if $\partial r / \partial r_0 \rightarrow 0$ and $r_0(r, z)$ becomes multivalued. If this happens, the density will diverge and the fluid picture will break down. All these features, if present, would be indicative of a wave breaking phenomenon. Breaking might be responsible for conversion of energy from macroscopic fluid modes into microscopic kinetic activity.

SIMULATIONS

Given an initial beam profile it is possible to evaluate the density n using Eq. (3). We write the initial cross sectional beam density at $z = 0$ in a general parabolic form $n_0(r_0) = \rho_h + \chi \rho_i(r_0)$, where the inhomogeneity parameter $0 \leq \chi \leq 1$, $\rho_h \equiv N_t / (\pi r_b^2)$, $\rho_i(r_0) \equiv \rho_h (2r_0^2 / r_b^2 - 1)$, and r_b is the beam radius. To diminish the effects arising from the pure envelope oscillations — Gluckstern resonances — we fix $r_b = \sqrt{K/\kappa}$, thus fixing the beam radius while the fluid description remains valid.

The compressibility factor exhibits a fast oscillatory motion accompanied by a slow secular growth, the latter arising as a result of spatial derivatives applied on the phase factors of Eq. (2). This means that given enough time, the compressibility will always cross its zero axis for any finite value of χ , resulting in a density divergence.

To further explore the significance of the diverging density, we have performed fully self-consistent N-particle simulations in the continuum limit; in practical terms we use $N_t = 10000$ in the simulations along with Gauss law. Taking advantage of the azimuthal beam symmetry and the Gauss law, a particle located at a position r experiences a field generated by the particles within a circle of radius r [6, 10]. Within the simulation, the trajectory of each particle is, therefore, also governed by Eq. (1) — unlike the fluid elements, however, particles are allowed to bypass one another.

In Fig. 1 we display the particle phase-space ($r, v \equiv r'$) for $\chi = 0.6$. The first panel (a) shows the initial distribution at $z = 0$ — all particles are still. In panel (b), after various propagating wave cycles, the system is about to build up an infinite density; velocity is still a single valued function of the space coordinate but a singularity (cusp) is forming. The third panel (c) shows the system at a time slightly larger than the first wave breaking time. Velocity

ceases to be a single valued function of the space coordinate — while some of the particles go through the wave others do not. This latter class of particles is accelerated by the wave front and forms a thin azimuthally symmetric jet or finger seen in the figure. High energy jet particles can reach far outside the beam core and may be very detrimental to the beam transport. The process shown in panel (c) repeats itself many times, see panel (d), as the system evolves toward a final stationary state and the previously unoccupied extensions of the phase-space are gradually filled with particles whose velocities are considerably larger than velocities in the beam core. After some time a stationary state is reached in which the beam separates into a cold dense core and a hot and extended halo of ejected particles, panel (e). Time evolution of the emittance [7] $\varepsilon \equiv 2\sqrt{\langle r^2 \rangle \langle v^2 \rangle - \langle r v \rangle^2}$, where $\langle \rangle$ denotes an average over particles, is shown in the last panel (f). At the wave breaking emittance suffers a sharp rise, followed by a rapid relaxation to the final stationary state in which large amplitude fluctuations subside. As mentioned, the

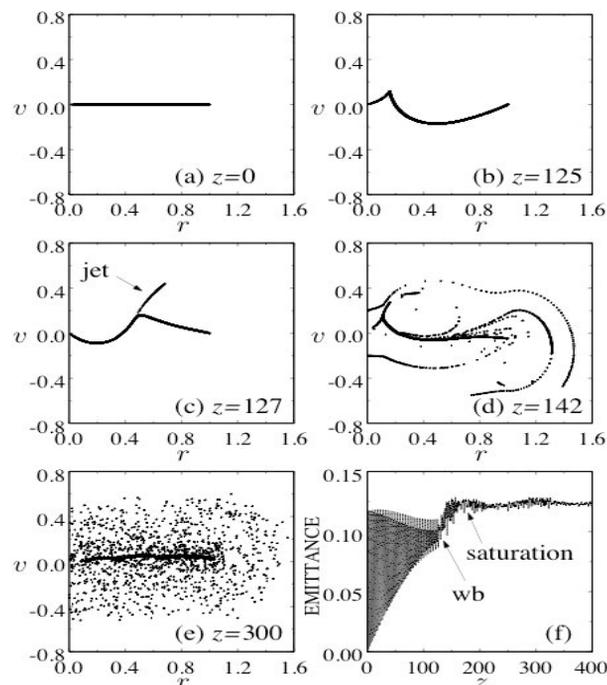


Figure 1: Time evolution in the phase-space (r, v) for $\chi = 0.6$. r is measured in units of $(K/\kappa)^{1/2}$, v in units of $K^{1/2}$, and z in units of $\kappa^{-1/2}$: panel (a) — initial condition at $z = 0$; panel (b) — after many wave cycles, but just before wave breaking; panel (c) — just after wave breaking, an azimuthal jet is expanding over the phase-space; panel (d) — while the first jet moves in the phase space new jets are being ejected; panel (e) — a phase portrait of a relaxed state; panel (f) — emittance growth leading to a final relaxation. Note that emittance saturates soon after the wave breaking (wb).

dominant mechanism for emittance growth is the singular build up of density followed by the wave breaking and jet production. In passing we note that the time of the first wave breaking depicted in the panel (c) of Fig. 1 agrees

well with the time when the compressibility factor $\partial r/\partial r_0$ obtained from the Lagrange fluid equations goes to zero. Our next goal is then to precisely calculate the instant at which the wave breaking takes place.

We first note that for an inhomogeneous density profile, each fluid element oscillates at a different frequency — rigid oscillations are possible only when the density profile across the beam cross section is homogeneous. Thus, nearby fluid elements will oscillate around their points of equilibria, slowly moving out-of-phase. At some point, however, two nearby fluid elements will overlap one another leading to a singular build up of density. When this happens the fluid picture will lose its validity and will have to be replaced by the full kinetic description given in terms of the Vlasov equation. The wave breaking occurs when the separation between any two fluid elements vanishes, $r(r_0 + \delta r_0, z) - r(r_0, z) \rightarrow 0$, for some value of r_0 . This is precisely equivalent to our condition for the appearance of a singular density, $\partial r/\partial r_0 \rightarrow 0$. Considering only the term linear in amplitude of Eq. (2) and neglecting the purely oscillatory term, as compared to the secular one, the time of breaking is found to be

$$z_{wb} \approx \min_{r_0} \left| \frac{1}{2\sqrt{Q}} \frac{\partial Q/\partial r_0}{A\partial\omega/\partial r_0} \right|; \quad (4)$$

breaking occurs whenever $\partial\omega/\partial r_0 \neq 0$. This is the case for all inhomogeneous particle beams. Unlike other systems in which one must have strong enough electric fields [11], here any sort of inhomogeneity leads to the wave breaking. As soon as the wave breaking takes place, particles with the same velocity as the density wave will be captured by the wave and surf down its front gaining kinetic energy (Fig. 1). Minimization in Eq. (4) can be performed

$$z_{wb} = \left(\frac{3}{2\kappa} \right)^{1/2} \frac{\alpha^3 (4\sqrt{1-\chi} + \chi - 1)}{(\sqrt{3} - \alpha)^2 (\sqrt{1-\chi} + \chi - 1)}, \quad (5)$$

where $\alpha \equiv (1 + 2\sqrt{1-\chi} - \chi)^{1/2}$.

In the limit $\chi \rightarrow 0$, the wave breaking time diverges as $z_{wb} \approx 81\sqrt{2}/\chi^3$. The subsequent breaks, however, occur on a much shorter time scale $\sim 1/\omega_0$ - we note that due to periodicity, after the first break every cycle brings the compression factor to a vanishing value again. Therefore, z_{wb} should also give us a good estimate of the relaxation time for the entire dynamics. In Fig. 2 we compare the wave breaking time obtained using the N particle dynamics simulation described above with z_{wb} given by Eq. (5). The figure reveals an amazingly good agreement between the two results. In the same plot we also show the relaxation time — defined as the time when the emittance first reaches its plateau value, see Fig. 1 (f). As expected, for smaller values of χ the time of relaxation follows closely the wave breaking time. For larger values of χ a deviation between the two data sets is observed.

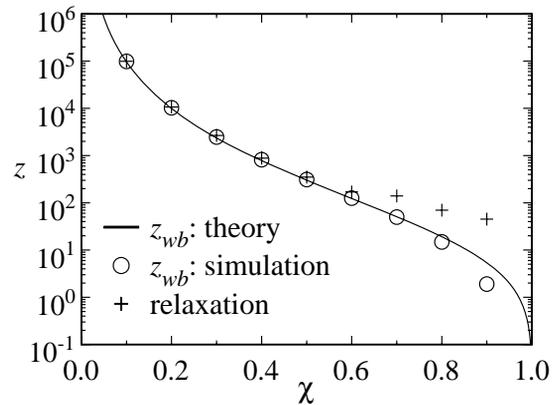


Figure 2: Comparison of the predicted time of the first wave breaking Eq.(5) (solid line), with the result of dynamics simulations (circles). Time is measured in units of $\kappa^{-1/2}$. A very good agreement between the theory and the simulations extends all the way to $\chi = 0.8$.

FINAL REMARKS

The theory presented in this paper allows to calculate precisely the time at which the wave breaking will take place. It also gives a very good estimate for the time of relaxation to the final stationary state. Unlike other systems in which the wave breaking occurs only when thresholds on driving fields are exceeded [11], inhomogeneous beams are found to be always unstable [12] and the wave breaking is unavoidable.

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