ON SKEW NONLINEAR RESONANCE IN THE SPRING-8 STORAGE RING

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Abstract

Recently we accomplish the matrix formulation for the canonical perturbation theory of the linear betatron coupling resonance in circular accelerators [1, 2]. By merging the perturbation theory with the matrix formalism, we manifest the symplectic structure of the former theory, and conversely derive the analytical representation for the latter. The formulation for the coupled betatron motion explicitly implies that the linear coupling causes the excitation of skew resonances through nonlinear fields with mid-plane symmetry [3, 4]. The third order skew coupling resonance is observed in the SPring-8 storage ring, for example, as the blow-up of the vertical beam size. For the purpose of studying the impact of the linear coupling on the skew nonlinear resonance, we investigate the characteristic behavior of the nonlinear resonance deduced from the matrix formulation of the perturbation theory.

IMPACT OF LINEAR COUPLING ON NONLINEAR RESONACE

We consider a nonlinear dynamics in a circular accelerator with small skew quadrupole distortions distributed over the circumference of a circular accelerator. The Hamiltonian describing the system consists of the three parts

$$H = H_0 + H_1 + H_3, \tag{1}$$

where H_0 is the Hamiltonian of unperturbed linear motion

$$H_0 = \frac{1}{2}\left(p_x^2 + p_y^2 + G_x x^2 + G_y y^2\right), \tag{2}$$

with the strengths of the quadrupole field $G_{x,y}$ giving the focusing force, and H_1 is the perturbing term for linear coupling

$$H_1 = Kxy, \tag{3}$$

with the strength of the skew quadrupole field K, and H_3 is the normal sextupole potential

$$H_3 = \frac{1}{3!} S\left(x^3 - 3xy^2\right) \tag{4}$$

with the strength of the sextupole field S. It is clear that, if it were not for the linear coupling term H_1, there is no skew sextupole resonance. Hence we show that the linear coupling gives rise to the skew sextuple potential from the normal one by rotating the coordinate system.

Linear coupled motion can be solved analytically by the perturbation treatment [5, 6]. On the other hand, the matrix formulation for the linear coupled system reveals the symplectic structure [7, 8]. If we merge both the theories, we can analytically represent the symplectic structure of the coupled system, which has been accomplished in [1, 2]. The matrix formulation gives the analytical representation of the symplectic rotation matrix, which converts the coordinate system from the physical one to the normal.

Now consider the linearly coupled system described by the Hamiltonian $H_0 + H_1$. Let the normal coordinates be $\vec{U} = (u, p_u, v, p_v)$. The matrix formulation gives the symplectic rotation matrix relating the physical coordinates $\vec{X} = (x, p_x, y, p_y)$ to the normal [7, 8]:

$$\vec{U} = \mathbf{T}\vec{X}, \tag{5}$$

where

$$\mathbf{T} = \begin{pmatrix} \tau I & -T^* \\ T & \tau I \end{pmatrix} \tag{6}$$

with $\tau^2 + \mathrm{Det}\,(T) = 1$. Here I is the 2-by-2 identity matrix and the symbol $*$ represents the adjoint operation $T^* = -ST^tS$ with $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Although the symplectic rotation matrix is given in terms of the transfer matrix, we cannot analytically deduce the dynamical behavior near the resonance. Using the perturbation theory with the single resonance approximation, we can solve the equation of motion analytically [5, 6]. Hence applying the matrix formulation to the perturbation theory, we derive the analytical representation for the symplectic rotation matrix [1, 2].

It is well-known that the solution of the unperturbed motion described by the Hamiltonian H_0 is given by

$$z_0(s) = a_{z0}\psi_z(s) + c.c., \tag{7}$$

$$p_{z0}(s) = a_{z0}\psi_z'(s) + c.c., \tag{8}$$

where ψ_z for $z = x, y$ are the unperturbed betatron motion

$$\psi_z(s) = \sqrt{\frac{\beta_z(s)}{2}}e^{i\phi_x(s)}, \tag{9}$$

with the betatron function $\beta_z(s)$ and the phase $\phi_z(s) = \int_{s_0}^{s} ds'/\beta_z(s')$, and a_{z0} some constants. Allowing the constants a_{z0} to vary, we can approximately solve the equation of coupled motion near the linear coupling difference resonance ($\nu_x - \nu_y \approx \text{integer}$) as

$$z(s) = a_z(s)\,\psi_z(s) + c.c., \tag{10}$$

$$p_z(s) = a_z(s)\,\psi_z'(s) + c.c., \tag{11}$$

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and

$$a_x(s) = A_1 e^{-2\pi i \nu_1 s/L} + A_2 e^{-2\pi i \nu_2 s/L}, \quad (12)$$

$$a_y(s) = \frac{C}{2}\left(\frac{A_1}{\nu_2} e^{2\pi i \nu_2 s/L} + \frac{A_2}{\nu_1} e^{2\pi i \nu_1 s/L}\right), \quad (13)$$

with the integration constants $A_{1,2}$ and the circumference L. Furthermore the tune deviation due to the coupling $\nu_{1,2}$ are given by

$$\nu_{1,2} = \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + |C|^2}\right), \quad (14)$$

where $\Delta = \nu_x - \nu_y - q$ is the distance from the linear difference resonance with an integer q, and C the coupling driving term

$$C = \frac{1}{2\pi}\int_0^L ds K(s)\sqrt{\beta_x(s)\beta_y(s)} \times e^{i\left[\phi_x(s)-\phi_y(s)-\frac{2\pi s}{L}(\nu_x-\nu_y-q)\right]}. \quad (15)$$

Using the solution derived by the perturbation theory, we can give the analytical representation for the symplectic rotation matrix [1, 2]

$$\mathbf{T} = \begin{cases} \sqrt{\frac{\nu_1}{\nu_1-\nu_2}}\begin{pmatrix} I & -\frac{1}{2\nu_1}N \\ -\frac{1}{2\nu_1}N & I \end{pmatrix} & \text{for } \Delta \geq 0 \\ \sqrt{\frac{-\nu_2}{\nu_1-\nu_2}}\begin{pmatrix} I & -\frac{1}{2\nu_2}N \\ -\frac{1}{2\nu_2}N & I \end{pmatrix} & \text{for } \Delta < 0 \end{cases}$$

with

$$N = E_x^{-1}\left[-\mathrm{Re}(C)\, I + \mathrm{Im}(C)\, S\right] E_y, \quad (16)$$

where $\mathrm{Re}(C)$ $[\mathrm{Im}(C)]$ indicates the real [imaginary] part of C and E_z is the normalization matrix

$$E_z = \begin{pmatrix} 1/\sqrt{\beta_z} & 0 \\ \alpha_z/\sqrt{\beta_z} & \sqrt{\beta_z} \end{pmatrix}. \quad (17)$$

In order to investigate the higher order resonance dynamics, we should consider the system in the normal coordinate where the eigen tunes are defined. Applying the transformation $\vec{X} = \mathbf{T}^{-1}\vec{U}$ to the third order Hamiltonian H_3, we obtain the sextupole potential in the normal coordinate. Expanding the potential with respect to the coupling driving term perturbatively, we find the skew sextupole potential $\tilde{H}_{3(1)}$ appearing in the next leading term:

$$\tilde{H}_{3(1)} = \frac{S}{4}\sqrt{\frac{\pm\nu_{1,2}}{\beta_x\beta_y(\nu_1-\nu_2)^3}} \times \left[\mathrm{Re}(C)\left\{\beta_x v^3 - (\beta_x + 2\beta_y)u^2 v\right\} + \mathrm{Im}(C)\left\{\alpha_x\beta_y v^3 - (\alpha_y\beta_x + 2\alpha_x\beta_y)u^2 v + \beta_x\beta_y\left(p_v u^2 - p_u uv - p_v v^2\right)\right\}\right]. \quad (18)$$

On the other hand, the leading term $\tilde{H}_{3(0)}$ is, off course, the normal sextupole potential. Note that, if it were not for the linear coupling, there exists no skew sextupole potential.

MEASUREMENTS OF THE NONLINEAR DIFFERENCE RESONANCE

For the purpose of confirming the validity of the present treatment, we study the skew nonlinear coupling at the SPring-8 storage ring, a synchrotron radiation light source for x-ray experiments. The major beam parameters of the SPring-8 storage ring are as follows. The beam energy is 8 GeV, and the operation point in user time $(40.15, 18.35)$, and the natural emittance 3.4 nmrad.

The skew sextupole coupling resonance $(2\nu_x - \nu_y \approx 62)$ is observed at the SPring-8 storage ring. Hence we investigate the response of the nonlinear resonance to the strength of the linear coupling resonance C. Varying the horizontal betatron tune with keeping the vertical one fixed $(\nu_y = 18.35)$, we perform the tune survey under the two conditions $|C| = 0.0012$ and 0.012. The smaller strength of the linear coupling resonance is achieved by the coupling correction using the skew quadrupole magnets.

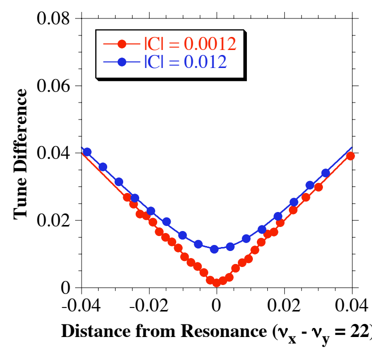

Figure 1: Tune difference in the vicinity of the linear coupling resonance.

Before investigating the nonlinear resonance, we explain the status of the linear coupling resonance. Figure 1 shows the measured data of the difference of the betatron tunes near the linear coupling resonance $(\nu_x - \nu_y \approx 22)$. As well-known, the minimum difference of the betatron tunes gives the strength of the coupling resonance, and the above mentioned values of the coupling strength are thus estimated. The analytical formulae for the projection beam size derived by the perturbation theory are [1, 2, 5, 6]

$$\sigma_x^2 = \frac{\Delta^2 + \frac{1}{2}|C|^2}{\Delta^2 + |C|^2}\beta_x\varepsilon_0, \quad (19)$$

$$\sigma_y^2 = \frac{\frac{1}{2}|C|^2}{\Delta^2 + |C|^2}\beta_y\varepsilon_0, \quad (20)$$

where ε_0 is the natural emittance. Hence the beam profile, especially the vertical beam size, is sensitive to the coupling resonance. In Fig. 2 we show the experimental results of the beam sizes at the tune survey. At the SPring-8 storage ring we measure the beam size by means of the visible light interferometer [9] and the x-ray imager [10]. The data detected by the former are represented by the open circles,

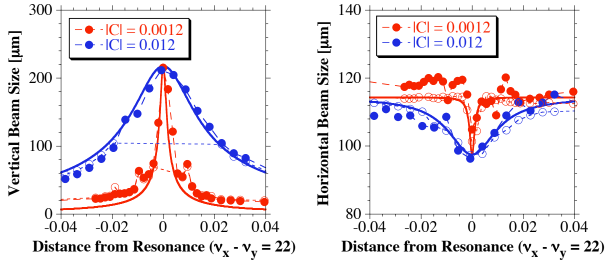

Figure 2: Vertical (left) and horizontal (right) beam sizes in the vicinity of the linear coupling resonance.

and those by the latter are depicted by the closed ones. The solid lines reflect the beam sizes expected by the analytical formula for the two cases of the coupling resonance. The analytical formulae of the beam sizes well describe the behavior of the beam profile in the vicinity of the linear deference resonance except for the sidebands observed in the case of the smaller coupling. The sidebands are excited by the synchrotron motion through the chromaticity or the vertical dispersion at sextupole magnets [11]. Note that the width of the resonance become narrow corresponding to the strength of the linear coupling.

Now we review the behavior of the beam sizes near the skew sextupole resonance shown in Fig. 3. The excitation of the nonlinear resonance is manifestly enhanced by the growth of the linear coupling. The global shift of the horizontal beam size is estimated to be the variation of the horizontal dispersion function. The solid lines represent

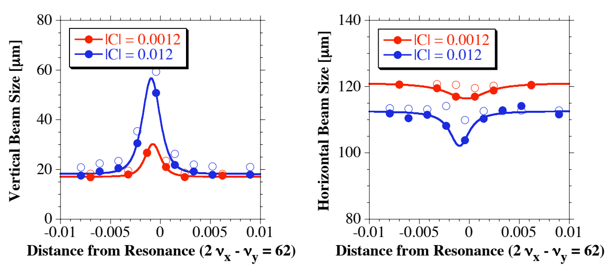

Figure 3: Vertical (left) and horizontal (right) beam sizes in the vicinity of the skew sextupole resonance.

the beam sizes supposed by the Lorentzian response on the distance from the resonance as for the linear coupling resonance described by Eqs. (19) and (20). The stronger the strength of the linear coupling resonance becomes, the larger not the width but the peak of the nonlinear resonance grows.

By inspecting the skew sextupole potential (18), we expect that the nonlinear coupling resonance is enhanced by approaching to the linear difference resonance. So we perform the tune survey under the different condition with fixed vertical tune $\nu_y = 18.20$. This change of the vertical tune reduces the distance of the skew sextupole resonance from the linear coupling one from 0.175 to 0.10. The strength of the nonlinear resonance is then enhanced by a factor of 1.74. The strength of the linear coupling resonance at this tune survey is estimated to be 0.012 same to the worse case of $\nu_y = 18.35$.

Figure 4 shows the result of the tune survey for the conditions of the different distances between the linear difference and the skew sextupole resonances. As expected, the

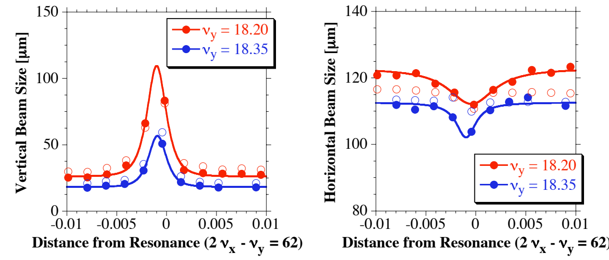

Figure 4: Vertical (left) and horizontal (right) beam sizes in the vicinity of the skew sextupole resonance with different distances from the linear difference resonance.

resonance excitation of the case $\nu_y = 18.2$ is stronger than that of $\nu_y = 18.35$.

SUMMARY

We elucidate that the linear coupling drives the skew nonlinear resonance through the normal sextupole magnets. This fact implies that the strength reduction of the linear coupling resonance leads to the decrease of the nonlinear resonance. This reduction of the nonlinear resonance is experimentally confirmed at the SPring-8 storage ring.

The linear coupling used to be corrected to reduce the vertical beam size, which is the figure of merit of the high brilliance of the light source ring. At the SPring-8 storage ring the linear coupling correction hardly improves the vertical beam size since the operation point is enough far from the linear resonance. Nevertheless, the correction of the linear coupling is still important because of weakening the nonlinear resonance.

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