SHORT QUADRUPOLE PARAMETRIZATION

R. Baartman, D. Kaltchev, TRIUMF, Vancouver, Canada *

Abstract

The Enge function can be used to parametrize any element with well-defined edges. If an element is too short, however, there is no unambiguous definition of the effective edge. We first demonstrate that very little fringe field detail is needed to obtain accurate maps even up to fifth order. Then we go on to show a simple fitting algorithm that works well for short as well as long quadrupoles. The results are true whether the quads are magnetic or electrostatic.

INTRODUCTION

Let us characterize the quadrupole strength along the axis s as $k(s)$. A long quadrupole is one in which second and higher derivatives of k are effectively zero at the quad centre. Under these conditions, it is a good approximation to model the quad as 3 regions: entrance fringe field, ideal quad body, exit fringe field. Virtually all beam transport codes take this approach [`GIOS`, `TRANSPORT`, `COSY-`∞, `TRACE-3D`], but they take different approaches in parametrizing the fringe fields. `GIOS` (and PSI-TRANSPORT[3]) use fringe field integrals to set up a matrix that is applied at the effective edges of the quad[1]. `COSY-`∞ depends upon a fit of the fringe field to an Enge function[2]; it first backtracks a drift from the effective quad edge, then Runge-Kutta integrates through the Enge function, backtracks an ideal quad to the effective edge, tracks through the ideal quad, and then reverses the procedure to exit the quad, finally arriving at the exit effective edge.

If a quadrupole's length is comparable or less than its aperture, we call it a short quad. This can be stated more quantitatively: it is short if the second derivative, k'', at the centre ($s = 0$) is not small compared with $k(0)/L^2$, where L is the length.

For short quads, both the matrix fringe field approach and the Enge-fit approach are not usable because there is no plateau in the strength function k. It is tempting to simply treat $k(0)$. However, the fringe field function would then have discontinuous second and higher derivatives. Attempting a fit to the Enge function results in diverging coefficients.

However, if we reduce the number of Enge coefficients to 1, there is a unique fit, and the map is sufficiently accurate for designing and modeling beamlines.

* TRIUMF receives funding via a contribution agreement through the National Research Council of Canada.

SOME THEORETICAL CONSIDERATIONS

First order

If one desires only a first order calculation, it is still important to properly characterize the fringe field because it is not possible to account for it by simply adjusting the effective length. In principle, by symmetry, there are only 2 parameters needed for each of x and y. (Each 4-element 2×2 matrix has unity determinant and by symmetry, the diagonal elements are equal.) One can adjust the effective length and strength, but fudging in this way would require different effective parameters for x and y: the focusing direction would need a shorter effective length, and the defocusing direction, longer. Roughly, the effect is to weaken the focusing by the fractional amount

$$\frac{\Delta f}{f} \approx \frac{1}{2}\frac{a^2}{fL} \tag{1}$$

where a is the aperture radius and L the effective length. Often, the effect is tiny because both f and L are much larger than the aperture. But low-energy secondary (muon, kaon) channels are examples of large-acceptance, short, strongly focusing lines where the effect can be larger than 10%.

A matrix approach is used in `GIOS`, where the effect of the fringe field is applied with a transfer matrix whose elements depend upon various "fringe field integrals"[1]. But it is over-specified, using 3 fringe-field integrals where only 2 parameters are in principle needed. As well, the fringe-field matrices in `GIOS` are not symplectic. Irwin's [4] matrix is symplectic, but uses only one parameter (I_1, which can be thought of as a normalized effective width of the fringe field) because $k(s)$ is assumed to be anti-symmetric about the edge, and so is not completely general. But as discussed below, it is sufficient for most purposes.

Third Order

Third order forces depend upon k' and k'' and so are singular in the hard edge limit. Nevertheless, the net effect through a quadrupole does not depend upon the extent of the fringe field: shorter fringe fields have stronger third order fields, but they act over shorter distances, and the two effects neatly cancel. This was shown in ref. [5]. There is, however, a small effect that comes into play when the quad is short compared with its length. When the derivatives are transformed away, the third order effect is found to be proportional to k^2 rather than k, and so the effective length of the third order is shorter than that of first order. In the `GIOS`

1-4244-0917-9/07/$25.00

approach, this is taken into account by fringe field integral I_4 (eqn. 7 below).

Fifth Order

Unlike third order, the fifth order elements of quadrupole transfer maps do depend directly on the extent of the fringe field. That is to say, they diverge in the hard-edge limit. However, the fifth order map depends mainly on the fringe field extent and not upon the details of its shape.

Enge Coefficients

The Enge function of the fringe field form is:

$$E(s) \equiv \frac{1}{1+\exp\left[\sum_{k=0}^{N-1} a_k(\frac{-s}{D})^k\right]} \qquad (2)$$

where D is the aperture diameter. COSY-∞ uses $N = 6$ coefficients. This is easily sufficient to give an accurate fit to a real quad. But is it necessary?

The Enge function has many drawbacks. For example, the series is not an expansion of the usual type where adding more terms refines the fit. On the contrary, adding one more term will require all the coefficients to change or else the effective edge will be shifted. As noted above, even up to fifth order, the detailed shape of the fringe field is not very important. Yet in order to get the effective length correctly reproduced to within 0.1% requires all 6 coefficients to be specified to at least 3 significant figures. So this is obviously not an efficient description.

Tanh Edge

For most cases of interest, the fringe field is relatively symmetric and one can set all Enge coefficients except a_1 to zero. In that case, the edge function becomes:

$$E(s) \equiv \frac{1}{1+\exp\left(\frac{-a_1 s}{D}\right)} = \frac{1+\tanh\left(\frac{a_1 s}{2D}\right)}{2} \qquad (3)$$

From this one can find the GIOS fringe field integrals:

$$I_1 \equiv \int^{s_b}\int E(s)dsds - s_b^2/2 = \frac{4\zeta(2)}{a_1^3} \qquad (4)$$

$$I_2 \equiv \int^{s_b} s\int E(s)dsds - s_b^3/3 = 0 \qquad (5)$$

$$I_3 \equiv \int^{s_b}\left(\int E(s)ds\right)^2 ds - s_b^3/3 = \frac{16\zeta(3)}{a_1^3} \qquad (6)$$

$$I_4 \equiv \int^{s_b} E^2(s)ds - s_b = -\frac{2}{a_1} \qquad (7)$$

where ζ is the Riemann zeta function.

This suggests an easy way to fit the fringe field to the one-parameter Enge function: Find $E(s)$ (for example by mapping the quad), numerically calculate I_4, set $a_1 = -2/I_4$. For the Enge coefficients built into COSY-∞ (these are from a family of PEP quads), I_4=-0.4328, so

Order	Rel. Error
1	0.0003
3	0.0014
5	0.055
7	0.35

Table 1: 6-parameter vs. 1-parameter Enge function comparison.

$a_1 = 4.62$. COSY-∞ results for the default Enge coefficients (0.296471 4.533219 -2.270982 1.068627 -0.036391 0.022261) are compared with the simplified case (0 4.62 0 0 0 0 0) in the table. (The maps are partially given in the Appendix.) As claimed, the approximation is very good up to and including fifth order. One can adjust a_1 to get fifth order term errors down to the level of 1% at the cost of raising the third order map error to 1%, but that would be cheating. These results apply to both magnetic and electrostatic quadrupoles.

SHORT QUADRUPOLES

A complete symmetric quad can be fitted to

$$k(s) = k_0\left[E(L/2+s) + E(L/2-s) - 1\right] \qquad (8)$$

Trivially, the integral $\int kds = k_0 L$. If the quad is long, $E(L/2) - 1 \ll 1$, so $k_0 = k(0)$, and then the effective length L is known as well from the integral. I_4 can be found unambiguously from eqn. 7.

On the other hand, if the quad is short, k_0 is not known. One can choose any value larger than $k(0)$ and still fit the other Enge coefficients. In the limit of the choice $k_0 = k(0)$ (a choice often naively made), the high order Enge coefficients diverge, because this represents the case where the second and higher derivatives of E(s) are discontinuous at $s = 0$.

A simple alternative is to use the one-parameter Enge function eqn. 3. This corresponds to the following fit.

$$k(s) = \frac{k_0}{2}\left[\tanh(a_1(L/2+s)) + \tanh(a_1(L/2-s))\right] \qquad (9)$$

This is in fact a 2-parameter fit, as k_0 and L are related by $\int kds = k_0 L$.

REFERENCES

[1] H. Matsuda and H. Wollnik "Third Order Transfer Matrices for the Fringing Field of Magnetic and Electrostatic Quadrupole Lenses" NIM **103**, p. 117 (1972).

[2] M. Berz "Computational Aspects of Design and Simulation: COSY INFINITY" NIM **A298**, p. 473 (1990).

[3] U. Rohrer "Compendium of Transport Enhancements", http://people.web.psi.ch/rohrer_u/trancomp.htm.

[4] J. Irwin, Chun-Xi Wang, "Explicit soft fringe maps of a quadrupole" Proc. of PAC95, p. 2376.

[5] R. Baartman, "Intrinsic Third Order Aberrations in Electrostatic and Magnetic Quadrupoles" Proc. PAC97, p. 1415.

APPENDIX: TRANSFER MAPS

The following are the COSY-∞ procedure and resulting transfer maps, edited for clarity. The two cases are: 6-parameter Enge function, and 1-parameter Enge function.

```
PROCEDURE RUN ;
VARIABLE Lq 1 ;VARIABLE Vq 1 ;
VARIABLE ap 1 ; VARIABLE S 1 ;
Lq:=0.4064;Vq:= .1;ap:=0.0516;
OV 7 2 0 ;
RP 1 30 1 ;
WRITE 69 'Default:␣6␣Enge␣Coeff.s';
UM ;
 fr 3;
mq Lq Vq ap ;
Pm 69 ;
WRITE 69 'Simplified:␣1␣Enge␣Coeff.';
UM ;
 loop s 1 2 1 ;
 fc 2 s 1 0. 4.62 0 0 0 0 ;
 endloop ;
 fr 3;
mq Lq Vq ap ;
Pm 69 ;
   ENDPROCEDURE ;
```

```
Default: 6 Enge Coeff.s
 0.80373 -0.92879  0.00000  0.00000  1000
 0.00000  0.00000  1.20979  1.07193  0010
 0.03117 -0.48531  0.00000  0.00000  3000
-0.22328 -2.40738  0.00000  0.00000  1020
 0.00000  0.00000 -0.16142 -1.91380  2010
 0.00000  0.00000  0.19450 -1.04865  0030
-0.56495 -2.50754  0.00000  0.00000  5000
-6.47894 -25.6781  0.00000  0.00000  3020
-4.51921 -19.3359  0.00000  0.00000  1040
 0.00000  0.00000 -2.66172 -13.6025  4010
 0.00000  0.00000 -5.70145 -36.3399  2030
 0.00000  0.00000 -0.98442 -4.53718  0050
-19.4869 -58.0318  0.00000  0.00000  7000
-329.422 -1189.99  0.00000  0.00000  5020
-451.688 -2204.89  0.00000  0.00000  3040
-130.313 -932.286  0.00000  0.00000  1060
 0.00000  0.00000 -122.169 -457.437  6010
 0.00000  0.00000 -470.009 -2217.53  4030
 0.00000  0.00000 -397.234 -2488.04  2050
 0.00000  0.00000 -27.0717 -223.553  0070
-----------------------------------------
Simplified: 1 Enge Coeff.
 0.80376 -0.92905  0.00000  0.00000  1000
 0.00000  0.00000  1.20983  1.07162  0010
 0.03114 -0.48465  0.00000  0.00000  3000
-0.22324 -2.40734  0.00000  0.00000  1020
 0.00000  0.00000 -0.16155 -1.91396  2010
 0.00000  0.00000  0.19460 -1.04973  0030
-0.53589 -2.35115  0.00000  0.00000  5000
-6.15146 -24.0411  0.00000  0.00000  3020
-4.29197 -18.4463  0.00000  0.00000  1040
 0.00000  0.00000 -2.52817 -12.8080  4010
 0.00000  0.00000 -5.32083 -34.4545  2030
 0.00000  0.00000 -0.92228 -4.34041  0050
-14.6395 -40.5916  0.00000  0.00000  7000
-242.061 -826.842  0.00000  0.00000  5020
-324.174 -1585.82  0.00000  0.00000  3040
-89.3586 -708.561  0.00000  0.00000  1060
 0.00000  0.00000 -92.4758 -328.478  6010
 0.00000  0.00000 -345.005 -1598.81  4030
 0.00000  0.00000 -281.155 -1847.34  2050
 0.00000  0.00000 -18.4175 -174.262  0070
-----------------------------------------
```


1-4244-0917-9/07/$25.00