SCALING LAWS FOR CROSSING NONLINEAR SPACE CHARGE RESONANCES*

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Abstract

Intrinsic fourth order space charge resonances may occur in linear as well as in circular accelerators. The fourth order difference resonance ("emittance exchange" or "Montague" coupling-resonance) and the fourth order structure resonance lead to emittance degradation depending on the strength of space charge, the crossing rate through the resonance and the lattice. Based on fully self-consistent simulation with a 2D code in the coasting beam limit, we present scaling laws for the "Montague" resonance and for the fourth order structure resonance in terms of simple power law expressions.

INTRODUCTION

Space charge driven resonances are an important issue in high intensity linear or circular accelerators. They can lead to emittance degradation or beam loss and cannot be eliminated by correction of the optics as they are intrinsic. While numerous studies have been dedicated to different aspects of this subject we focus in this paper on two candidates of fourth order: The "Montague" emittance coupling resonance [1] and the fourth order structure resonance caused by the systematic "space charge octupole". The aim of our study is to extract scaling laws describing in a simplified way the effect of space charge, tune ramp, emittances etc. on the expected emittance degradation. Although both candidates reveal enough complexity, involving self-consistent space charge and all the specifics of a fourth-order resonance, we have been encouraged by the findings of a recent detailed study on the Montague case [2]. Based on analytical theory and self-consistent simulation it was found there that the emittance change is the same if the ratio of the square of the space charge tune shift and the tune ramp is invariant. This is confirmed here also for the fourth order structure resonance and used as basis for deriving our self-consistent scaling laws.

We note here that in a non-self-consistent approach (using a fixed space charge potential with an update of rms quantities during the evolution of the resonance) the fourth order structure resonance, along with other lower and higher order space charge driven resonances, has recently been the subject of a parameter study, with application to the FFAG and other proton drivers [3]. One of the important findings of Ref. [3] has been that the lattice effect can be factorized and thus separated from the space charge and tune ramp effect. The scaling law in Ref. [3] employs, however, lower powers of space charge tune shift and tune ramp than we find in the present paper, which may be attributed to the effect of self-consistency.

Our self-consistent simulations are carried out with the MICROMAP-code [4] using 50.000 particles and a 128x128 grid with conducting boundary conditions on a square box of width 6 times the horizontal rms size of the beam. We use the lattice of the SIS18 at GSI, which has 12 super-periods with the option of a triplet focusing cell per period. In our simulations we vary emittances, currents and working points over a broader range than normally used and take Gaussian initial distributions. Since the Montague-resonance is driven by the zeroth harmonic of the lattice functions, e.g. $2Q_x - 2Q_y \approx 0$, it is expected to be "immune" to the actual lattice beta function, which is also confirmed by earlier simulation [2]. For the fourth order structure resonance, $4Q_y \approx 12$, the weight of the underlying Fourier harmonic matters and the answer is lattice dependent.

MONTAGUE SCALING

The basic idea is to start from analytical expressions for stop-band width and growth times, which were derived in Ref. [2]. The width of the stop-band in terms of a spread of the horizontal tune was found analytically as:

$$\Theta = \frac{3}{2}(\sqrt{\epsilon}_r - 1)\Delta Q_x, \tag{1}$$

where ΔQ_x is the incoherent space charge tune spread defined as maximum tune spread of a Gaussian beam, i.e. twice the KV-equivalent tune shift; ϵ_r is the ratio of initial transverse emittances (here assumed ≥ 1, without loss of generality). The number of betatron periods, after which emittance exchange occurs, was obtained in Ref. [2] approximately as (including also the limit $\epsilon_r \rightarrow 1$):

$$N_{ex}^{-1} \approx (\sqrt{\epsilon}_r - 1)\frac{\Delta Q_x}{Q_{0,x}}. \tag{2}$$

As an example for dynamical crossing we move the working point $Q_{0,x}$ upwards, starting from the side of lower tunes, over the range $5.15 \leq Q_{0,x} \leq 5.27$ enclosing the stop-band, while $Q_{0,y}$ is kept fixed at the value 5.21. For this crossing "from below"we apply a linear tune ramp in time. A picture of the final emittances after crossing the

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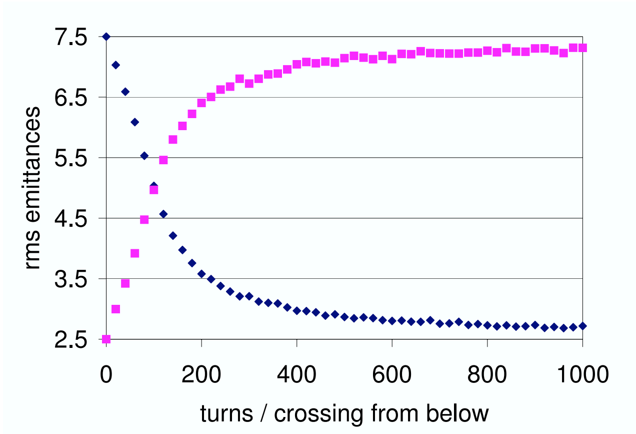

Figure 1: Final emittances after crossing the stop-band at variable number of turns ($\Delta Q_y = -0.105$).

band at variable number of turns is shown in an example of Fig. 1.

In the 100 turns case the essential part of the stop-band, which has a tune width of 0.04, is crossed in 33 turns or 205 betatron periods. This time well agrees with the fastest rise time of the static case, which therefore sets the time-scale needed for a crossing at a speed chosen to just equalize final emittances, in agreement with Eq. 2.

For very slow crossing emittances are practically exchanged. For faster crossing the exchange is only partial, with a linear dependence on the number of turns as suggested by Fig. 1, hence proportional to the inverse of the tune ramping rate.

Our scaling is based on the observation confirmed by simulation (see cases in Table 1) that the emittance growth is unchanged, if for a given ϵ_r the actual width of the stop-band - $\propto \Delta Q_x$ - is crossed during a time, which is inversely proportional to ΔQ_x. Hence, this two-fold dependence on ΔQ_x justifies the assumption - also confirmed by simulation - that the emittance growth depends on the scaling term

$$\frac{(\Delta Q_x)^2}{\dot{Q}}, \tag{3}$$

where $\dot{Q}$ is defined as tune change in x per turn.

Using Eq. 3 and Eqs. 1,2 for the additional dependence on ϵ_r, we therefore suggest a scaling law for the relative growth of the smaller emittance (here in y)

$$\frac{\Delta\epsilon_y}{\epsilon_y} = \alpha_{2,2}\frac{((\sqrt{\epsilon_r}-1)\Delta Q_x)^2}{\dot{Q}}, \tag{4}$$

assuming that the full stop-band is crossed at constant rate and $\dot{Q}$ is the change of the horizontal tune change per turn (ignoring here the sign of it). $\alpha_{2,2}$ is a factor, which needs to be determined; a first estimate follows from the graph of Fig. 1 as $\alpha_{2,2} \approx 0.5$.

To ensure the validity of Eq. 4 we use it to determine the corresponding $\alpha_{2,2}$ for a broader diversity of simulations

Table 1: Montague simulations

case	ϵ_x	ϵ_y	$-\Delta Q_x$	turns	$\dot{Q}$	$\frac{\Delta\epsilon_y}{\epsilon_y}$	$\alpha_{2,2}$
1	7.5	2.5	0.058	50	0.0024	0.36	0.48
2	7.5	2.5	0.029	100	0.0006	0.37	0.48
3	7.5	2.5	0.059	15	0.0081	0.12	0.52
4	7.5	2.5	0.059	100	0.0012	0.81	0.53
5	7.5	2.5	0.028	400	0.0003	0.72	0.52
6	10.0	2.0	0.045	100	0.0012	1.31	0.51
7	7.5	3.8	0.072	100	0.0012	0.45	0.60

and summarize results in Table 1 (rms emittances in units of π mm mrad): This table shows the validity of the proposed scaling and confirms that $\alpha_{2,2} \approx 2$ is a good choice for the fore-factor. In case 1 the $Q_{0,x}$ was swept over the interval 4.15...4.27, in case 2 over 4.18...4.24 ($Q_{0,x} = 4.21$), which confirms the invariance claimed in Eq.3. In all other cases the sweep was over 5.15...5.27 ($Q_{0,x} = 5.21$). Cases 3-5 show the validity of the scaling with the speed of crossing. Cases 6 and 7 show different emittance ratios; for too small ratios the scaling appears to slightly underestimate the emittance exchange effect.

Crossing in the opposite direction (downwards tune ramp) leads to a typically 20 to 30% weaker emittance exchange due to the fact that space charge tune shifts (with progressive emittance exchange) counteract the applied tune ramp; for very slow crossing this asymmetry is even more pronounced (details see Ref. [2]).

FOURTH ORDER STRUCTURE RESONANCE SCALING

This purely space charge driven resonance appears near the condition $4Q_{0,y} = N$, with N the number of super periods; likewise in a transport lattice, where the phase advance per cell of 90^0 is approached from above by means of space charge or by reducing the focusing. It was first predicted theoretically in Refs. [6],[7]. The fourth order structure resonance can be in competition with the envelope instability, which appears under similar conditions (see also a more recent study in Ref. [8]).

In Fig. 2 we show a phase space plot obtained for a simulation of the SIS18 lattice. In normal operation the vertical phase advance per cell (and super-period) is close to 100^0, hence $Q_{0,y} \approx 3.3$. For our example we have chosen a downwards tune ramp crossing symmetrically the value $Q_{0,y} = 3$ by starting from 3.3 and ending at 2.7 in all cases considered. Parameters are those of case 2 in Table 2 below. The scatter plots show the typical result of particles trapped in fourth order resonance islands, which move to larger amplitudes to compensate the lowered bare tune by weaker space charge. At the end of the ramp the amplitude dependent de-tuning results in a diffuse halo in vertical phase space.

In order to find an appropriate scaling law we have simulated a set of parameter combinations as shown in Ta-

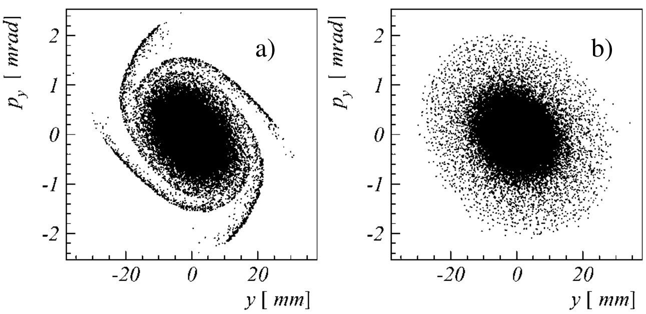

Figure 2: a) Vertical phase space plot exactly on resonance, b) at end of tune ramp.

ble 2 in the order of increasing emittance growth (initial rms emittances equal in x, y). Cases 5,7,8 and 10 show

Table 2: Fourth order mode simulation

case	ϵ_y	-ΔQ_y	turns	$\dot{Q}$	$\frac{\Delta\epsilon_y}{\epsilon_y}$	α_4
1	7.5	0.20	50	0.01200	0.08	0.0076
2	7.5	0.05	1600	0.00038	0.12	0.0027
3	7.5	0.10	400	0.00150	0.12	0.0028
4	7.5	0.20	100	0.00600	0.13	0.0028
5	7.5	0.20	150	0.00400	0.20	0.0020
6	7.5	0.05	3200	0.00019	0.34	0.0019
7	7.5	0.20	200	0.00300	0.36	0.0020
8	7.5	0.20	300	0.00200	0.79	0.0020
9	7.5	0.10	1600	0.00038	1.21	0.0017
10	7.5	0.20	400	0.00150	1.26	0.0018
11	2.5	0.20	400	0.00150	1.27	0.0018
12	7.5	0.30	225	0.00267	1.81	0.0016
13	7.5	0.20	800	0.00075	2.95	0.0010
14	7.5	0.10	3200	0.00019	2.97	0.0010
15	7.5	0.30	450	0.00133	3.79	0.0008

that the emittance growth is best fitted by a scaling $\propto \dot{Q}^{-2}$. Comparing cases 4,3 as well as 3,2 and 10,9 suggests that the emittance growth is only a function of the scaling term introduced in Eq. 3 (here taken in y); also, independence of the actual emittance value is shown by comparing cases 10 and 11. Cases with $< 20\%$ emittance growth don't fit into this scaling - presumably due to transient effects. We thus find that the results of Table 2 are best fitted by using the power 2 of the scaling term, hence we obtain

$$\frac{\Delta\epsilon_y}{\epsilon_y} = \alpha_4 \left(\frac{(\Delta Q_x)^2}{\dot{Q}} \right)^2 . \tag{5}$$

This expression is then applied to simulation results to solve for the fore-factor α_4 as shown in Table 2. This suggests that $\alpha_4 \approx 0.0018$ as average for cases 5-12 is an appropriate choice for it. For emittance growth exceeding 200% (cases 13-15) this scaling overestimates - presumably due to saturation effects.

It should be noted that the downwards tune ramp considered so far has a stronger effect on the emittance growth than the equivalent upwards tune ramp due to the fact that the shrinking space charge tune shift pertaining to a decreasing charge density - due to the resonance effect - counteracts the downwards moving tune. In this way, particles trapped into resonance islands are driven to larger amplitudes, which decreases the charge density and helps maintain the resonance condition due to the space charge detuning effect. This effective "tune ramp deceleration" is changed into a "tune ramp acceleration"for the upwards tune ramp. This asymmetry effect between downwards and upwards is negligible, if the downwards tune ramp leads only to small emittance growth (of the order of or less than 10%); but for large downwards effect the upwards emittance growth is found to be reduced to about 30% or less in most cases.

A remark is appropriate on the influence of the initial distribution. A Gaussian one provides a strong "space charge octupole" (which rolls of outside of the beam as opposed to a real octupole), but even an initially uniform KV beam would develop a similar nonlinearity via the fourth order instability studied in Ref. [7].

SUMMARY

Our scaling relationships show that only few parameters are needed to predict the effect of resonance crossing on rms emittances for the two fourth order resonances considered in the frame of a fully self-consistent simulation. In both cases the emittance increment depends on a power of the scaling term $\frac{\Delta Q_x^2}{\dot{Q}}$, which is the main parameter. For given tune ramp $\dot{Q}$ the emittance growth is found to depend on the second power of the space charge tune shift for the Montague case, and on the fourth power for the fourth order structure resonance. The growth is practically independent of the lattice in the Montague case. For the fourth order structure resonance the periodic variation of the beta function is essential; the dependence of α_4 on the focusing structure (triplet or doublet, where the latter enhances the resonant effect) will be studied in a forthcoming paper.

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