EXTENSION OF NAPOLY INTEGRAL FOR TRANSVERSE WAKE POTENTIALS TO GENERAL AXISYMMETRIC STRUCTURE

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Abstract

The Napoly integral for the wake potential calculations in the axisymmetric structure is a very useful method because the integration of E_z field can be confined in a finite length instead of the infinite length by deforming the integration path, which reduces CPU time for the accurate calculations. However, his original method could not be applied to the transverse wake potentials in a structure where the two beam tubes on both sides have unequal radii. In this case, the integration path needs to be a straight line and the integration stretches out to an infinite in principle. We generalize the Napoly integrals so that integrals are always confined in a finite length even when the two beam tubes have unequal radii, for both longitudinal and transverse wake potential calculations. The extended method has been successfully implemented to ABCI code.

INTRODUCTION

Calculating wake potentials is an important issue in the design of accelerators. Napoly *et. al.*[1, 2] originally developed the method to calculate wake potentials, where the integration along the longitudinal direction only comes from the path across the cavity gap. This simplification is essential for the numerical calculations, for instance, for large structures, for short bunches *et. al.* to reduce computer memory and CPU time.

However, in their technique the radii of the chamber must be equal on both sides of the structure, for the calculation of wake potentials of higher than the dipole mode. In this paper, we generalize their method to the case that both sides of the structure can be unequal for any mode.

THE BASIC CONCEPT AND PREPARATIONS

The integration of vector along the closed path is zero, when the one-form which is defined by the vector is closed. This feature is very useful to deform the path of integration, calculating wake potentials. Following Napoly *et. al.*[2], we introduce one-form and represent some identities to prepare for calculating wake potentials in the next section.

Let us consider the axisymmetric cavity as shown in Fig.1. We use the cylindrical coordinate (r, θ, z). From now on, we assume that the radius of the beam tube of the downstream side a_{out} is smaller than that of the upstream side a_{in}, for simplicity (This constraint is removed in the final expression of the wake potential, i.e. Eq.(25)).

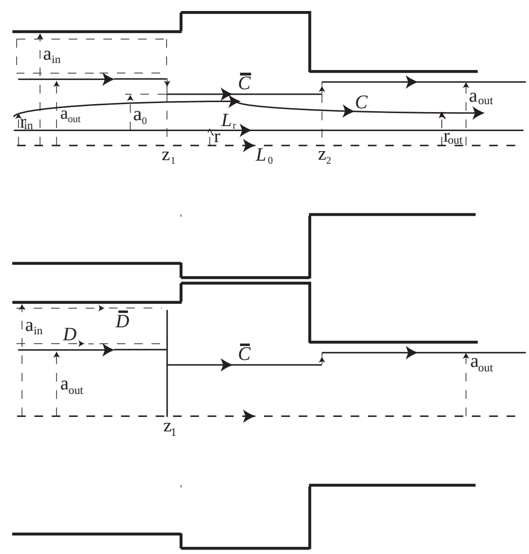

Figure 1: The cavity and the integration path

Given $\vec{r}_0 = (r_0, \theta_0 = 0)$, which is the transverse coordinates of the source particle, the electromagnetic fields are expanded as follows,

$$(E_r, B_\theta, E_z)(r, \theta, z, t) = \sum_{m=0}^{\infty} (e_r, b_\theta, e_z)^{(m)}(r, z, t) \cos m\theta, \tag{1}$$

$$(B_r, E_\theta, B_z)(r, \theta, z, t) = \sum_{m=1}^{\infty} (b_r, e_\theta, b_z)^{(m)}(r, z, t) \sin m\theta. \tag{2}$$

The θ-dependence of the solutions is the consequence of the azimuthal symmetry of the system. The electromagnetic fields can be decomposed as $E = E^{(s)} + E^{(r)}$ and $B = B^{(s)} + B^{(r)}$, where $(E^{(s)}, B^{(s)})$ are the source fields and $(E^{(r)}, B^{(r)})$ are the radiated fields (which are the solutions of homogeneous Maxwell equations).

Let us introduce the following definition for a generic field:

$$\bar{\phi}(z, s) \equiv \phi(z, t(z, s)), \tag{3}$$

in such a way that,

$$\partial_z \bar{\phi}(z, s) = (\partial_z + \partial_{ct})\phi(z, t(z, s)), \tag{4}$$

where s is the distance behind a given origin $z_0 = ct$ in the exciting bunch, and $t(z, s) = (z + s)/c$.

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By using this notation, we denote the closed one-forms as,

$$S^{(m)} = (r^m[\bar{e}_r^{(r)} + c\bar{b}_\theta^{(r)} - \bar{e}_\theta^{(r)} + c\bar{b}_r^{(r)}]^{(m)}, r^m[\bar{e}_z^{(r)} + c\bar{b}_z^{(r)}]^{(m)}), \tag{5}$$

$$D^{(m)} = (r^{-m}[\bar{e}_r^{(r)} + c\bar{b}_\theta^{(r)} + \bar{e}_\theta^{(r)} - c\bar{b}_r^{(r)}]^{(m)}, r^{-m}[\bar{e}_z^{(r)} - c\bar{b}_z^{(r)}]^{(m)}), \tag{6}$$

in (r, z)-plane [2]. We should notice that these one-forms are composed of only radiated fields.

In order to deform the integration contour which appears in the wake potential, the several useful relations should be introduced in advance. First of all, we need the asymptotic expression of the radiated fields for $z \to \pm\infty$. If we assume that the source field is defined by,

$$\phi^{(s)}(r, \theta, z, t) = \frac{Q}{2\pi\epsilon_0}\lambda(s)\left[\log\left(\frac{a_{out}}{r_>}\right) + \sum_{m=1}^{\infty}\frac{\cos m\theta}{m}\left(\frac{r_<}{r_>}\right)^m\right], \tag{7}$$

$$A_z^{(s)}(r, \theta, z, t) = \phi^{(s)}(r, \theta, z, t)/c, \tag{8}$$

where $r_> = sup(r, r_0)$ and $r_< = inf(r, r_0)$, the radiated fields for $z \to \pm\infty$ are given by,

$$\lim_{z\to-\infty}\bar{e}_r^{(r)}(r, z, s) = -\lim_{z\to-\infty}\bar{e}_\theta^{(r)}(r, z, s) = \lim_{z\to-\infty}c\bar{b}_r^{(r)}(r, z, s) = \lim_{z\to-\infty}c\bar{b}_\theta^{(r)}(r, z, s) = \frac{Q}{2\pi r\epsilon_0}\lambda(s)\sum_{m=1}^{\infty}\left(\frac{rr_0}{a_{in}^2}\right)^m, \tag{9}$$

$$\lim_{z\to-\infty}\bar{e}_z^{(r)}(r, z, s) = \lim_{z\to-\infty}c\bar{b}_z^{(r)}(r, z, s) = 0, \tag{10}$$

$$\lim_{z\to\infty}\bar{e}_r^{(r)}(r, z, s) = -\lim_{z\to\infty}\bar{e}_\theta^{(r)}(r, z, s) = \lim_{z\to\infty}c\bar{b}_r^{(r)}(r, z, s) = \lim_{z\to\infty}c\bar{b}_\theta^{(r)}(r, z, s) = \frac{Q}{2\pi r\epsilon_0}\lambda(s)\sum_{m=1}^{\infty}\left(\frac{rr_0}{a_{out}^2}\right)^m, \tag{11}$$

$$\lim_{z\to\infty}\bar{e}_z^{(r)}(r, z, s) = \lim_{z\to\infty}c\bar{b}_z^{(r)}(r, z, s) = 0. \tag{12}$$

Applying these expressions to the loop integration of Eqs.(5) and (6), one can obtain the following identities:

$$\int_{\mathcal{L}_r} D^{(m)}(r, z, s)dl = \int_{\mathcal{C}} D^{(m)}(r', z', s)dl', \tag{13}$$

$$\int_{-\infty}^{\infty}\left[\bar{e}_z^{(r)}(r, z, s)\right]^{(m)}dz = -\int_{-\infty}^{\infty}\left[c\bar{b}_z^{(r)}(r, z, s)\right]^{(m)}dz - \frac{Q\lambda(s)r_0^m r^m}{m\pi\epsilon_0}\left(\frac{1}{a_{in}^{2m}} - \frac{1}{a_{out}^{2m}}\right), \tag{14}$$

$$\int_{\bar{\mathcal{D}}} S^{(m)}[a_{in}, z, s]dl = \int_{\mathcal{D}} S^{(m)}[a_{out}, z, s]dl - \int_{a_{out}}^{a_{in}} drr^m\frac{2Q}{\pi r\epsilon_0}\lambda(s)\left(\frac{rr_0}{a_{in}^2}\right)^m + \int_{a_{out}}^{a_{in}} drr^m\left[\bar{e}_r^{(r)} + c\bar{b}_\theta^{(r)} - \bar{e}_\theta^{(r)} + c\bar{b}_r^{(r)}\right]^{(m)}\Big|_{z=z_1}, \tag{15}$$

$$\int_{\bar{\mathcal{D}}} D^{(m)}[a_{in}, z, s]dl = \int_{\mathcal{D}} D^{(m)}[a_{out}, z, s]dl + \int_{a_{out}}^{a_{in}} drr^{-m}\left[\bar{e}_r^{(r)} + c\bar{b}_\theta^{(r)} + \bar{e}_\theta^{(r)} - c\bar{b}_r^{(r)}\right]^{(m)}\Big|_{z=z_1}, \tag{16}$$

where r_{in} and r_{out} are the end radii of the contour $\mathcal{C}$ expressed in the upper figure of Fig.1 and the contour $\bar{\mathcal{D}}$ and $\mathcal{D}$ are described in the lower figure of Fig.1. The positions z_1 and z_2 specify the contour $\bar{\mathcal{C}}$ denoted in Fig.1.

When the contour $\mathcal{D}$ is chosen on the axis of the tube in Eq.(16), we obtain another relation:

$$\int_{-\infty}^{z_1} dzc\left[\bar{b}_z^{(r)}\right]^{(m)}\Big|_{r=a_{in}} = -\int_0^{a_{in}} drr^m\frac{2Q}{\pi r\epsilon_0}\frac{\lambda(s)}{a_{in}^m}\left(\frac{rr_0}{a_{in}^2}\right)^m + \int_0^{a_{in}} dr\frac{r^m}{a_{in}^m}\left[\bar{e}_r^{(r)} + c\bar{b}_\theta^{(r)} - \bar{e}_\theta^{(r)} + c\bar{b}_r^{(r)}\right]^{(m)}\Big|_{z=z_1}. \tag{17}$$

By using Eqs.(15)-(17), we find that the integration of e_z over z between $-\infty$ and z_1 can be replaced as,

$$\frac{2}{a_{out}^m}\int_{-\infty}^{z_1} dz\left[e_z\right]^{(m)}\Big|_{r=a_{out}} = \left(\frac{1}{a_{out}^{2m}} - \frac{1}{a_{in}^{2m}}\right) \times \int_0^{a_{in}} drr^m\left[e_r + cb_\theta - e_\theta + cb_r\right]^{(m)}\Big|_{z=z_1} + \frac{2Q}{m\pi\epsilon_0}\lambda(s)r_0^m\left(\frac{1}{a_{out}^{2m}} - \frac{1}{a_{in}^{2m}}\right) - \int_{a_{out}}^{a_{in}} dr\frac{r^m}{a_{out}^{2m}}\left[e_r + cb_\theta - e_\theta + cb_r\right]^{(m)}\Big|_{z=z_1} - \int_{a_{out}}^{a_{in}} dr\frac{1}{r^m}\left[e_r + cb_\theta + e_\theta - cb_r\right]^{(m)}\Big|_{z=z_1}. \tag{18}$$

We should notice that the integration from $-\infty$ to z_1 is confined in the finite length in Eqs. (17) and (18).

THE WAKE POTENTIAL FOR $M \geq 1$

The expression for the $m = 0$ longitudinal wake potential was already derived by Napoly *et al* themselves [1, 2]. This expression can be applied to the calculation of the wake potential for any type of structure. They also derived the wake potentials for $m \geq 1$, only when the radii of the chamber was equal on both sides of the structure. In this

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section, we generalize the expression of the $m(\neq 0)$-th order longitudinal wake potential $W_z^{(m)}(r,\theta,s)$ to the case that the two beam tubes have unequal radii.

The longitudinal wake potential is defined as,

$$W_z^{(m)}(r,\theta,s) \equiv -\frac{1}{Q}\int_{-\infty}^{\infty} dz E_z(r,\theta,z,t(z,s)), \quad (19)$$

where Q is total charge, which is characterized by a longitudinal charge distribution $\lambda(s)$ normalized as,

$$\int_{-\infty}^{\infty}\lambda(s)ds = 1. \quad (20)$$

The transverse wake potential, which is defined as,

$$W_\perp^{(m)}(r,\theta,s) \equiv -\frac{1}{Q}\int_{-\infty}^{\infty} dz(E_\perp + v\times B)(r,\theta,z,t(z,s)), \quad (21)$$

can be derived from the longitudinal one by applying the Panofsky-Wenzel theorem [3]:

$$\partial_s W_\perp^{(m)}(r,\theta,s) = \nabla_\perp W_z^{(m)}(r,\theta,s). \quad (22)$$

Since the longitudinal wake potential can be rewritten by the integration of the radiated field $[\bar{e}_z^{(r)}]^{(m)}$, we can easily deform the integration path by using the property of the closed one-forms. By using Eqs.(13) and (14), Eq.(19) is described as follows,

$$\begin{aligned} W_z^{(m)}(r,\theta,s) &= -\frac{\cos m\theta}{2Q} r^m \times \\ &\left[\int_{\bar{\mathcal{C}}} D^{(m)}(r',z',s)dl' + \frac{1}{a_{out}^{2m}}\int_{\bar{\mathcal{C}}} S^{(m)}(r',z',s)dl'\right] \\ &= -\frac{\cos m\theta}{2Q}\frac{r^m}{a_{out}^m}\times \\ &\left\{\int_{\bar{\mathcal{C}}} dz'\left[\bar{e}_z^{(r)}\left(\frac{a_{out}^m}{r'^m}+\frac{r'^m}{a_{out}^m}\right) - c\bar{b}_z^{(r)}\left(\frac{a_{out}^m}{r'^m}-\frac{r'^m}{a_{out}^m}\right)\right]^{(m)}\right. \\ &+ \int_{\bar{\mathcal{C}}} dr'\left[(\bar{e}_r^{(r)}+c\bar{b}_\theta^{(r)})\left(\frac{a_{out}^m}{r'^m}+\frac{r'^m}{a_{out}^m}\right)\right. \\ &\left.\left. +(\bar{e}_\theta^{(r)}-c\bar{b}_r^{(r)})\left(\frac{a_{out}^m}{r'^m}-\frac{r'^m}{a_{out}^m}\right)\right]^{(m)}\right\}. \quad (23) \end{aligned}$$

It is necessary to rewrite Eq.(23) by the real fields instead of the radiated fields. Since the source field is calculated by Eqs.(7) and (8), the longitudinal wake potential is expressed by

$$\begin{aligned} W_z^{(m)}(r,\theta,s) &= -\frac{\cos m\theta}{2Q}\frac{r^m}{a_{out}^m}\times \\ &\left\{\int_{\bar{\mathcal{C}}} dz'\left[e_z\left(\frac{a_{out}^m}{r'^m}+\frac{r'^m}{a_{out}^m}\right) - cb_z\left(\frac{a_{out}^m}{r'^m}-\frac{r'^m}{a_{out}^m}\right)\right]^{(m)}\right. \\ &+ \int_{\bar{\mathcal{C}}} dr'\left[(e_r+cb_\theta)\left(\frac{a_{out}^m}{r'^m}+\frac{r'^m}{a_{out}^m}\right)\right. \\ &\left.\left. +(e_\theta-cb_r)\left(\frac{a_{out}^m}{r'^m}-\frac{r'^m}{a_{out}^m}\right)\right]^{(m)}\right\}. \quad (24) \end{aligned}$$

In $\bar{\mathcal{C}}$-integration in Eq.(24), the longitudinal coordinate z substantially moves from $-\infty$ to z_2. This integration needs to be confined in a finite length. Actually, the component which comes from the path from the point $(r = a_{out}, z = -\infty)$ to $(r = a_{out}, z = z_1)$ can be replaced by the another expression by using Eq.(18). The general expression of the longitudinal wake potential (which can be applied to not only the case $a_{out} \leq a_{in}$ but also to the case $a_{out} \geq a_{in}$) is given by,

$$\begin{aligned} W_z^{(m)}(r,\theta,s) &= -\frac{\cos m\theta}{2Q} r^m \\ &\times\left\{\frac{Q}{\pi\epsilon_0}\lambda(s)\frac{r_0^m}{m}\left(\frac{1}{a_{out}^{2m}}-\frac{1}{a_{in}^{2m}}\right)\right. \\ &+\int_{z_1}^{z_2} dz\left(\frac{[e_z-cb_z]^{(m)}}{a_0^m}+\frac{a_0^m[e_z+cb_z]^{(m)}}{\min(a_{in}^{2m},a_{out}^{2m})}\right) \\ &+\left(\frac{1}{a_{out}^{2m}}-\frac{1}{a_{in}^{2m}}\right)\int_0^{a_0} dr' r'^m[e_r+cb_\theta-e_\theta+cb_r]^{(m)}\Big|_{z=z_j} \\ &+\int_{a_{in}}^{a_0} dr'\frac{[e_r+cb_\theta+e_\theta-cb_r]^{(m)}}{r'^m}\Big|_{z=z_1} \\ &+\int_{a_{in}}^{a_0} dr'\frac{r'^m[e_r+cb_\theta-e_\theta+cb_r]^{(m)}}{a_{in}^{2m}}\Big|_{z=z_1} \\ &+\int_{a_0}^{a_{out}} dr'\frac{[e_r+cb_\theta+e_\theta-cb_r]^{(m)}}{r'^m}\Big|_{z=z_2} \\ &\left.+\int_{a_0}^{a_{out}} dr'\frac{r'^m[e_r+cb_\theta-e_\theta+cb_r]^{(m)}}{a_{out}^{2m}}\Big|_{z=z_2}\right\}, \quad (25) \end{aligned}$$

where the radial size a_0, which specifies the contour $\bar{\mathcal{C}}$, is defined in the upper figure of Fig.1, the notation z_j is z_1 in the case of $a_{out} \leq a_{in}$ and that is z_2 in the case of $a_{out} \geq a_{in}$.

This formula has been implemented to ABCI code [4]. The usefulness is examined in the reference [5] in this conference.

SUMMERY

The integration path in the wake potentials can be deformed by using the closed one-form defined in (r,z)-plane. This procedure generalizes Napoly integral for any m and for any structure. The integration of E_z field over z in an infinite length is replaced by that of electromagnetic fields in a finite region (typically the cavity gap size).

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