NUMERICAL SOLVER WITH CIP METHOD FOR FOKKER PLANCK EQUATION OF STOCHASTIC COOLING*

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Abstract

A numerical solver for a Fokker-Planck equation in a stochastic momentum cooling process by using a CIP method is developed. The Fokker-Planck equation is separated to a coherent phase and an incoherent phase for a numerical calculation procedure. The incoherent term is written as the diffusion equation, while the coherent term is described by the advection equation. For solving the advection term of the equation, we use the CIP method, and the results are compared with other numerical schemes. It is found that the rational function based CIP method is effective as the numerical solver for the stochastic momentum cooling model.

INTRODUCTION

Stochastic momentum cooling is operated to obtain a high-density beam within a small momentum spread for experiments.

A Fokker-Planck equation is used as a powerful tool for investigating the stochastic momentum cooling process [1]. The cooling term is described as the advection term, and the diffusion term is occurred by the beam signal and the amplifier noises [2]. The Fokker-Planck equation can be numerically solved as the advection-diffusion equation.

The Constrained Interpolation Profile (CIP) method [3] is a useful scheme in the numerical calculation for the advection equation. Using the CIP method, we can numerically solve nonlinear equations including the advection term with less discretized grid numbers.

In this study we propose the Fokker-Planck equation solver by using the CIP method for the stochastic cooling model. The example calculations show good agreement for the last COSY experimental result [4]. The other solvers for the numerical simulation are compared for the particle conservation law.

SOLVER WITH CIP METHOD

Fokker-Planck Equation for Model of Stochastic Momentum Cooling

The simplified Fokker-Planck equation for a model of a stochastic cooling is given as [1]

$$\frac{\partial \Psi}{\partial t} + \frac{\partial}{\partial E}\left(F\Psi - D\frac{\partial \Psi}{\partial E}\right) = 0, \tag{1}$$

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where $\Psi \equiv \Psi(E,t) \equiv dN/dE$ is the distribution function, $F \equiv F(E)$ and $D \equiv D(\Psi(E),t)$ are the terms for the cooling force and diffusion coefficients, respectively. The coherent and incoherent terms are derived by the electrical characteristics of the real feedback system [2].

Semi-Lagrangian Approximation (Time Splitting to Advection and Non-Advection Phases)

Eq. (1) can be split into

$$\frac{\Psi^* - \Psi^n}{\Delta t} = -\Psi^n \frac{\partial F}{\partial E} + \frac{\partial}{\partial E}\left(D\frac{\partial \Psi}{\partial E}\right), \tag{2}$$

and

$$\frac{\Psi^{n+1} - \Psi^*}{\Delta t} + F\frac{\partial \Psi^*}{\partial E} = 0, \tag{3}$$

where Δt is the time width between $n+1$ and n time steps, Ψ^* indicates the value after the non-advection phase calculation, Ψ^{n+1} and Ψ^n are the values of Ψ at $n+1$ and n time steps. Eq. (2) is the non-advection equation, and Eq. (3) is the advection equation.

Discretization for Non-Advection Phase

Figure 1 shows the define positions for the physical values in discretized grids. Here i is index in the energy space

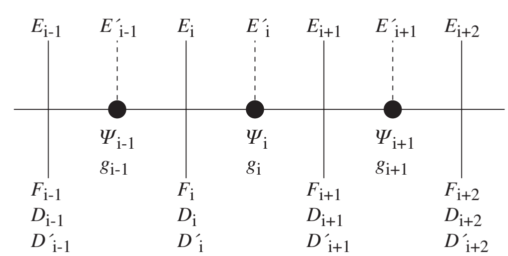

Figure 1: Grids used in this system.

discretized. We rewrite Eq. (2) as

$$\frac{\Psi_i^* - \Psi_i^n}{\Delta t} = -\Psi_i^n \left.\frac{\partial F}{\partial E}\right|_i + \left.\frac{\partial D'}{\partial E}\right|_i, \tag{4}$$

where $D' = D\,\partial\Psi/\partial E$. The discretization for the non-advection phase Eq. (4) is as

$$\frac{\Psi_i^* - \Psi_i^n}{\Delta t} = -\Psi_i^n \frac{F_{i+1} - F_i}{E_{i+1} - E_i} + \frac{D'_{i+1} - D'_i}{E_{i+1} - E_i}, \tag{5}$$

where $D'_i = D_i(\Psi_i^n - \Psi_{i-1}^n)/(E'_i - E'_{i-1})$ and $D'_{i+1} = D_{i+1}(\Psi_{i+1}^n - \Psi_i^n)/(E'_{i+1} - E'_i)$. The discretization for the non-advection phase is carried out by the explicit method for the time integration and the central difference scheme for the spatial derivative. Consequently, we calculate next equation for the time integration of the non-advection phase,

$$\begin{aligned}\Psi_i^* &= \Psi_i^n - \Delta t\,\Psi_i^n \frac{F_{i+1}-F_i}{E_{i+1}-E_i} + \frac{\Delta t}{E_{i+1}-E_i} \\ &\left(D_{i+1}\frac{\Psi_{i+1}^n - \Psi_i^n}{E'_{i+1}-E'_i} - D_i \frac{\Psi_i^n - \Psi_{i-1}^n}{E'_i - E'_{i-1}} \right). \quad (6)\end{aligned}$$

Discretization for Advection Phase

The CIP method [3] can be applied to solve the advection phase Eq. (3) as

$$\Psi_i^{n+1} = a_i\,\xi^3 + b_i\,\xi^2 + g_i^n \xi + f_i^n, \qquad (7)$$

$$g_i^{n+1} = 3a_i\,\xi^2 + 2b_i\,\xi + g_i^n, \qquad (8)$$

where g_i^n means the spatial derivative of Ψ_i^n, $\xi = -F\Delta t$, and the coefficients

$$a_i = \frac{g_i + g_{iup}}{\Delta^2} + \frac{2(\Psi_i^* - \Psi_{iup}^*)}{\Delta^3}, \qquad (9)$$

$$b_i = \frac{3(\Psi_{iup}^* - \Psi_i^*)}{\Delta^2} - \frac{2g_i + g_{iup}}{\Delta}, \qquad (10)$$

where

$$iup = \begin{cases} i+1 & (F \le 0), \\ i-1 & (F \ge 0), \end{cases} \qquad (11)$$

and $\Delta = E_{iup} - E_i$. After the calculation for the non-advection phase, the above calculations are operated for the advection phase.

Implimentation of Derivative Coefficient for Non-Advection Phase

We should calculate the non-advection phase of the derivative $g \equiv \partial\Psi/\partial E$. Eq. (1) is rewritten by

$$\frac{\partial g}{\partial t} + F\frac{\partial g}{\partial E} = \frac{\partial H}{\partial E} - g\frac{\partial F}{\partial E}, \qquad (12)$$

where

$$H = -\Psi\frac{\partial F}{\partial E} + \frac{\partial}{\partial E}\left(D\frac{\partial \Psi}{\partial E}\right). \qquad (13)$$

As mentioned in previous subsection, we can also split the equation into

$$\frac{g^* - g^n}{\Delta t} = \frac{\partial H}{\partial E} - g^n\frac{\partial F}{\partial E}, \qquad (14)$$

$$\frac{g^{n+1} - g^*}{\Delta t} + F\frac{\partial g^*}{\partial E} = 0. \qquad (15)$$

We compute next equation for the derivative of the non-advection phase Eq. (14),

$$\begin{aligned}g_i^* &= g_i^n + \left(\frac{\Psi_{i+1}^{*\prime} - \Psi_{i+1}^{n\prime}}{E_{i+1}-E_i} - \frac{\Psi_i^{*\prime} - \Psi_i^{n\prime}}{E_{i+1}-E_i}\right) \\ &- g_i^n \frac{F_{i+1}-F_i}{E_{i+1}-E_i}\Delta t. \qquad (16)\end{aligned}$$

Eq. (15) was already solved by Eq. (8) using the CIP method.

Calculation Procedure

By using the equations discussed previously the numerical calculation procedure is as follows;

1. Set initial conditions
2. Prepare diffusion coefficient D_i
3. Calculate Eq. (6)
4. Calculate Eq. (16)
5. Calculate advection phase by the CIP method
6. Repeat from 2. to 5. with time advances

SIMULATION EXAMPLES

We numerically simulate the stochastic momentum cooling process by using the procedures described in the previous section. The example calculation is performed by using the last COSY experimental data [4]. The condition is summarized as Table 1.

Table 1: Parameters for numerical simulations

Beam	
Momentum	3.224 GeV/c
Total energy	3.358 GeV
Kinetic energy	2.42 GeV
Particle number	10^{10}
Energy spread (1σ)	0.774 MeV
Ring	
Dispersion	-0.1
Momentum acceptance	+/- 1.5×10^{-3}
Stochastic cooling system	
Band width	$1 \sim 1.8$ GHz
Effective temperature	80 K
Electrode length	32 mm
Electrode width	20 mm
Gap height	20 mm
Impedance	50 Ω
Number pickup and kicker	24
TOF from pickup to kicker	0.3229 μsec
System delay	-0.04 ns

Figures 2 and 3 show the particle (proton) distributions in the calculated and experimental results. Figure 4 shows the momentum spread (1σ) during the cooling operation in the calculated and experimental results. It is confirmed that the numerical solver quite well represents the experimental data.

Figure 5 shows the particle number during the another cooling operation condition at each numerical scheme used for the advection term. The ”Up-Wind” label indicates the case by using the 1st-order up-wind method for the advection term. The ”CIP with MmB” and ”CIP” labels are for the solvers by using the CIP method with and without Maximum and Minimum Bonds (MmB) scheme [5]. The

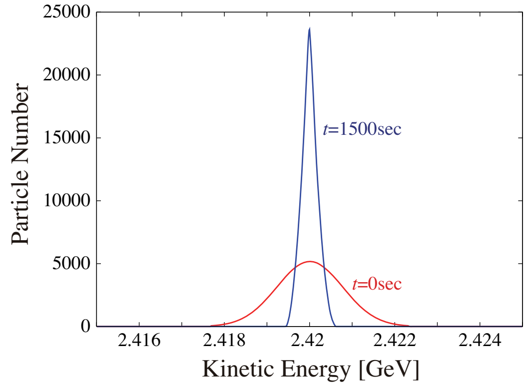

Figure 2: Simulation result for the system gain at 96 db.

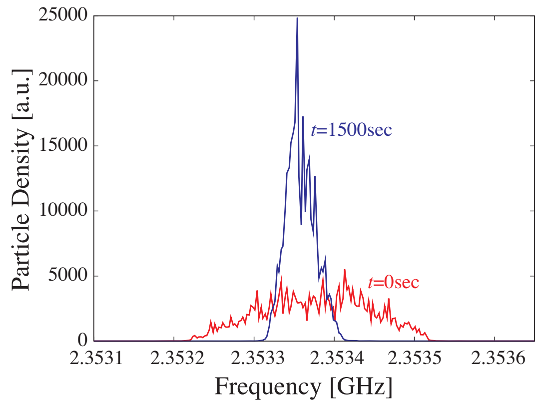

Figure 3: Experimental result [4] for same condition at Fig. 2.

"RCIP" shows the calculation result for the rational function based CIP (RCIP) method [6]. In the case of "Up-Wind", the conservation law is largely violated as shown in Fig. 5. The result for the CIP method gives an unphysical phenomena after the long time beam cooling, because the distribution function has the quite sharp distribution at the center due to the stochastic cooling. The solver using the

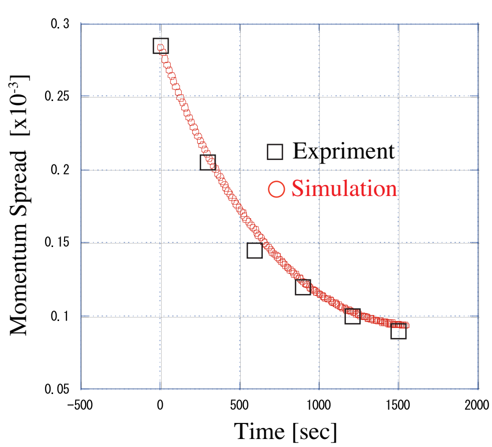

Figure 4: Momentum spread during the stochastic cooling.

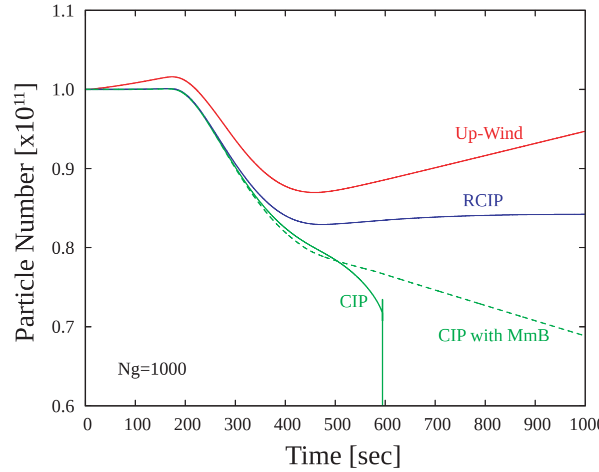

Figure 5: Particle number at each calculation method for the conservation law check.

CIP method with MmB also causes the unphysical particle losses after the long term operation. We can obtain the good result in the case for the RCIP method as shown in Fig. 5.

CONCLUSIONS

We proposed the Fokker-Planck equation solver based on the CIP method for the stochastic momentum cooling model. With this study, we have confirmed that the numerical solver quite well represents the experimental data. Using the RCIP method, the conservation law was maintained even in the calculations for the long term operation. Consequently, the developed numerical solver is a useful tool to investigate the stochastic momentum cooling process.

Using the developed solver, we will be able to predict quite accurately the stochastic cooling process of notch filter in the storage ring such as HESR of FAIR project [7] in our future work.

REFERENCES

[1] T. Katayama and N. Tokuda, Part. Accel. **21**, 99 (1987).

[2] T. Katayama, *private communications*.

[3] H. Takewaki and T. Yabe, J. Comput. Phys. **70**, 355 (1987).

[4] H. Stockhorst, T. Katayama, R. Stassen, R. Maier and D. Prasuhn, "Longitudinal Stochastic Cooling Simulations in Comparison with Cooling Experiments of COSY", Annual Report, IKP, Forshungzentrum, (2007).

[5] W. Huamo and Y. Shuli, "MmB - A new class of accurate high resolution schemes for conservation laws in two dimensions", IMPACT of Computing in Science and Engineering , Vol.1, Issue 3, September 1989, pp.217-259.

[6] F. Xiao, "Numerical Schemes for Advection Equation and Multi-layered Fluid Dynamics", Ph.D. Thesis, Tokyo Institute of Technology, (1996).

[7] HESR, "Baseline Technical Report", http://www.fz-juelich.de/ikp/hesr

