COLLIMATOR WAKEFIELDS: FORMULAE AND SIMULATION *

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Abstract

The effect of a leading particle on a trailing particle due to resistive and geometric wakefields in collimators can be described by expanding in a series of angular mode potentials $W_m(s)$. Several formulae for these are given in the literature. We compare these formulae with numerical predictions from codes that solve the EM field equations. We also show how the EM code results can be used to numerically obtain angular mode potentials suitable for use in tracking codes.

WAKE FIELDS IN COLLIMATORS

In the collimation system of the ILC the beams pass very close to the collimator edges, which may lead to significant wakefield effects. Collimators, unlike cavities, are not basically resonant structures, so the behaviour is not dominated by resonant modes. Long range inter-bunch wakefields are therefore unimportant, whereas short range intra-bunch effects can be significant. Because parts of the beam are near the edge of the conductor, high order angular modes can become important. Effects from transverse wakefields are believed to be more important than longitudinal ones. The study of these effects continues[1, 2].

BUNCH WAKES AND DELTA WAKES

A particle going through an aperture induces charges and currents which produce electromagnetic fields (wake fields) that act on later particles. The simulation programs need to know the effect that a leading particle with some transverse co-ordinate $\vec{r}\,'$ has on some particle with transverse co-ordinate $\vec{r}$ trailing by some distance s. For axially symmetric apertures the transverse dependence is taken care of by a sum over modes $m = 0, 1, 2..$ leaving the wake functions $W^m(s)$. These functions for individual particles are termed the *delta wakes.*

The effect on the trailing particle comes not only from this particular leading particle but from all particles in the bunch which are ahead of it. This aggregated wake functions are termed the *bunch wakes*.

Bunch wakes can be determined from delta wakes by summation or integration. The question addressed here is how to obtain knowledge of the delta wakes from the bunch wakes.

This is important because delta wakes (geometric and resistive) have been calculated only in some particular cases for particular apertures. There are programs such as GdfidL, MAFIA, and ECHO which solve Maxwell's equations on a grid for arbitrary aperture shapes and compute the bunch wake fields, assuming a Gaussian bunch of some specified size σ. But for particle simulation codes such as Merlin[3] needed to track the behaviour of the non-Gaussian bunch shapes that will be produced, the delta wake functions are needed.

As an example we consider a tapered collimator in which the beampipe of radius 19 mm tapers down to 2 mm over a distance of 5 cm, and then back out again.

The simple and widely used delta wake formula[4] for a beam pipe of radius a tapering in to an aperture of radius b

$$W^m_\delta(s) = 2\left(\frac{1}{a^{2m}} - \frac{1}{b^{2m}}\right) e^{-ms/a}\Theta(s) \qquad (1)$$

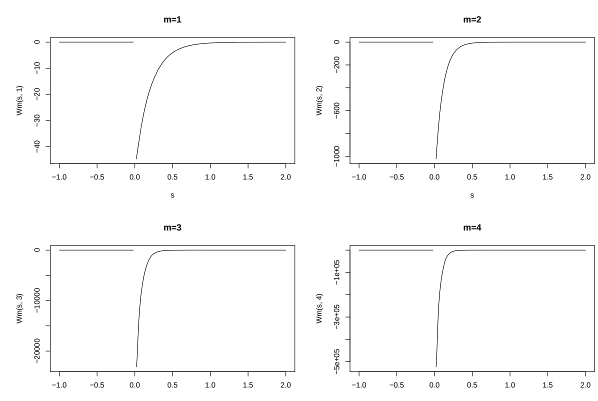

Figure 1: Transverseδ wakes suggested by Raimondi et al

and shown in the figure was justified on the grounds that it gives bunch wakes which agree with simulations from the MAFIA program. This agreement is clearly a necessary but not a sufficient condition, as there are various functions that could give similar acceptable agreement. We need to find a clear way of getting from the bunch to the delta wakes.

ECHO SIMULATIONS

This is provided through deconvolution. The bunch wake $W_b(s)$ is a convolution of the delta wake $W_\delta(s)$ and the Gaussian beam profile, $G(s;\sigma)$

We determined the bunch wakes for this aperture with the ECHO2D program [5]. A bunch of $\sigma = 0.05$ mm was used, with a z step of 0.005 mm.

To extract the delta wakes from the bunch wakes shown in Figure 2 we take the Fourier transform, divide by the Fourier Coefficients of a Gaussian with $\sigma = 0.05$, and transform back. The result - the top left plot in Figure 3 - is unphysical as it has large high frequency components.

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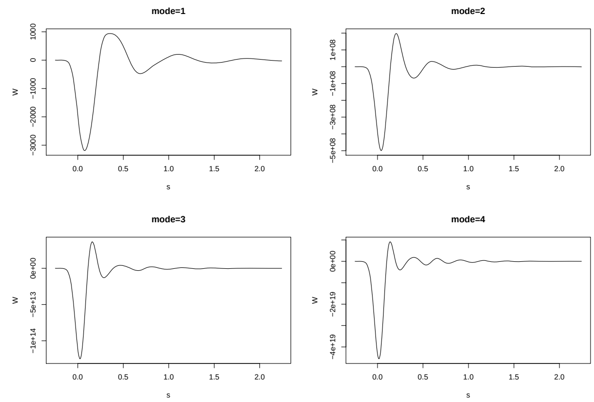

Figure 2: Bunch wakes from ECHO2D

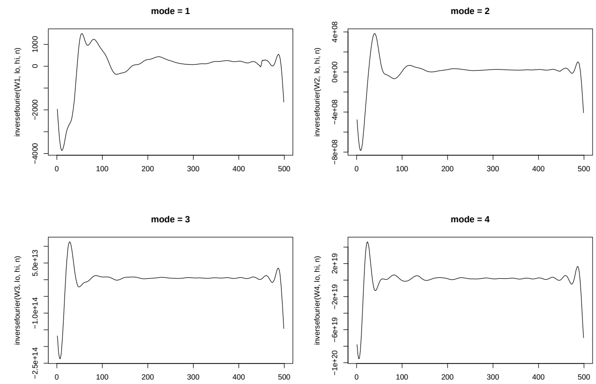

Figure 4: Reconstructed δ wakes with $\gamma = 10$

(It is not wrong: it gives the correct bunch wake when recombined with a Gaussian.) The problem is that at high frequencies small coefficients of the bunch wake are divided by very small coefficients of the Gaussian, and small fluctuations in the denominator have massive effects.

This problem can be removed by a simple inverse filter. If the magnitude of the Gaussian coefficient c_k is smaller than some tuning constant $1/\gamma$ the multiplication of the amplitude is capped at γ. The effects of different γ values are shown in Figure 3. We choose to work with $\gamma = 10$

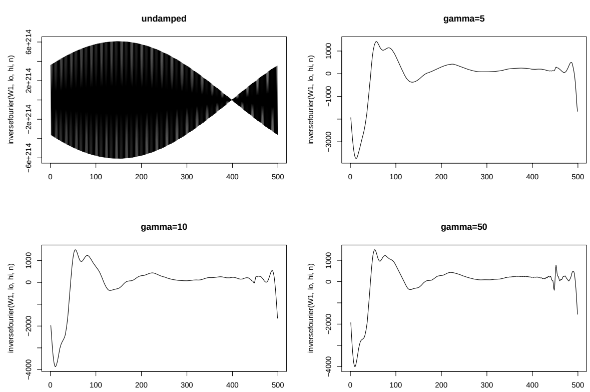

Figure 3: Reconstructed W_δ^1 with different damping

With this choice we reconstruct the δ wakes. Figure 4 corresponds to the bunch wakes in Figure 2.

There are clearly similarities to the proposed wakes of Figure 1 but also differences. The wakes become narrower and larger as the mode increases. However there are also differences: the reconstructed wakes show more structure (including crossing the axis to become positive) than expected.

Probing the same aperture with Gaussian pulses of different σ gives different bunch wakes, yet when deconvoluted (with the appropriate σ) the results are compatible.

Nevertheless there is structure at large s values which is clearly non-physical. It must arise because the function is periodic: it has a large step at $s = 0$ which is not easy for a Fourier transform to accommodate and the effects wrap around back through the origin. Given that this structure is clearly wrong raises a question about how much the other structure can be trusted.

The wake function must be zero for negative s. This is a basic causality requirement[6]: particles are only affected through wake fields of earlier particles. We must have $W_\delta(s) = 0$ for all $s < 0$. We therefore investigate how to include this knowledge into the Fourier deconvolution process.

USING THE CAUSAL NATURE

Suppose a region is mapped into $-\pi, \pi$ and a grid of $2N + 1$ points set up, $r = -N, ...0, ...N$, with $x = r\pi/N$. We are interested in functions for which $f(x_r) = f_r = 0 \forall (r < 0)$. We call these 'causal functions'.

The Fourier expansion may be written

$$f_r = b_0 + \sum_k a_k \sin(kr\pi/N) + \sum_k b_k \cos(kr\pi/N) \quad (2)$$

For any negative r we require $f_r = 0$ which means that the total contribution from the a_i must be equal and opposite to the contribution from the b_i, cancelling exactly. At the equivalent positive r the cosine terms are the same, whereas the sine terms all change sign. Therefore for r in the range 1 to N a causal function must have

$$\mathbf{Sa} = \mathbf{b_0} + \mathbf{Cb} \quad (3)$$

where $S_{jk} = \sin(kj\pi/N)$ etc and $\mathbf{b_0}$ is a vector of length N with all components equal to b_0. This is a relationship between the coefficients which must be satisfied for any causal signal.

The matrix $\mathbf{S}$ is square and symmetric and orthogonal. The off-diagonal members of $\mathbf{S}^2$ are all zero: the diagonal ones are all $\frac{N}{2}$ except for the final bottom right one, which is zero. However the coefficient a_N is meaningless as it multiplies $\sin(Nr\pi/N)$ which is zero for all points r. Hence we are justified in writing

$$\mathbf{a} = \frac{2}{N}\left(\mathbf{Sb_0} + \mathbf{SCb}\right) \quad (4)$$

The point f_{-N}, which maps onto the point f_N, does not depend on the a_k, as all $\sin(kN\pi/N)$ are zero. The cosines alternate between $+1$ and -1. Therefore to ensure this point is zero one must have

$$b_0 = b_1 - b_2 + b_3 - b_4 ... \quad (5)$$

So we have a procedure for working with Fourier transforms of causal signals. The cosine coefficients $b_j (j = 1....N)$ can be freely chosen or determined. The above equations 4 and 5 are then used to give the constant term b_0 and the sine coefficients a_j, and this uniquely guarantees that the function is zero for all negative r.

APPLICATION TO CONVOLUTION

If the (causal) signal with coefficients a_j, b_j, is convoluted with a smearing function (a Gaussian of known width σ) with coefficients α_j, β_j, then the bunch wake is

$$f_r = b_0\beta_0 + \sum_k (a_k\beta_k + \alpha_k b_k) S_{rk} + \sum_k (b_k\beta_k - a_k\alpha_k) C_{rk} \quad (6)$$

Using the above expressions for b_0 and $\mathbf{a}$, this can be written $\mathbf{f} = \mathbf{Qb}$ where

$$Q_{rj} = (-1)^j\beta_0 + S_{rj}\alpha_j + C_{rj}\beta_j + \frac{2}{N} \sum_k (S_{kr}\beta_k - C_{kr}\alpha_k) \left(\sum_\ell S_{k\ell}C_{\ell j} + (-1)^j \sum_\ell S_{k\ell}\right) \quad (7)$$

To determine the parameters by fitting the data d_r we adjust the coefficients such that $\chi^2 = \sum_{r=-N}^{N} (d_r - f_r)^2$ is minimised. This strategy is justified not only on grounds of convenience but as follows: we have N free parameters but more than N data values, so we will not get an exact fit. A general Fourier transform fits the f_r to the d_r perfectly, but if you curtail the series at some wavenumber $k_{max} < N$, which you may well do for convenience, the coefficients are actually those for which χ^2 is minimised.

Minimising $\chi^2 = (\mathbf{d} - \mathbf{Qb})^2$ leads to the equation

$$\tilde{\mathbf{Q}}\mathbf{d} = \tilde{\mathbf{Q}}\mathbf{Qb} \quad (8)$$

which can be solved to give the coefficients b_j, and the coefficients b_0 and a_j then obtained. The desired causal delta wake is also writable in matrix form as $\mathbf{w} = \mathbf{Mb}$, where $\mathbf{M}$ is readily obtained from Equations 2, 4 and 5.

If this method also gives unphysical high frequency components (and it seems to do so, though not as badly as the direct method) then this can be easily and naturally dealt with by including a regularisation term of $k = 2$ Tikhonov form[7]. A matrix $\mathbf{T}$ is introduced which produces the numerical second derivatives, and the quantity to be minimised is $\chi^2 + \lambda\tilde{\mathbf{b}}\tilde{\mathbf{M}}\tilde{\mathbf{T}}\mathbf{TMb}$. λ is adjustable: positive λ increases the smoothness at the expense of a (small) increase in χ^2. Values around $\lambda = 10^{-4}$ seem to give good results.

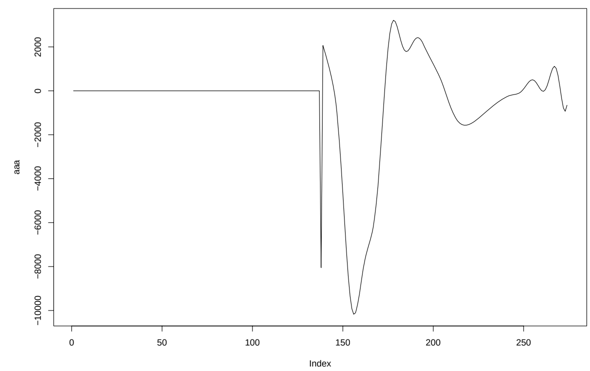

Figure 5: Reconstructed causal W^1_δ

Figure 5 shows the result. There are still some artefacts - the range had to be restricted for reasons of speed. But the main and medium-size features of the delta wake persist from that of Figure 4.

CONCLUSIONS

The deconvolution method looks to provide a way of extracting delta wakes from bunch wakes. Further tuning of the method is required, but already we can see that simple formulae, though they may be useful approximations, do not describe the full structure.

Not only can this method be used to validate formulae, the data values can be written to tables which will be used as desired in the simulation packages.

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