STERN-GERLACH FORCE ON A PRECESSING MAGNETIC MOMENT *

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Abstract

Use of the Stern-Gerlach force for attaining the spin-states separation of a particle beam is reconsidered in a new method where the magnetic moments are made to precess, at variance with a previously considered case where the magnetic moment conserves its direction in space.

GENERAL CONSIDERATIONS

The time varying Stern-Gerlach interaction of a relativistic fermion with an e.m. wave has been studied [1], particularly in the example of standing waves built up inside a rectangular TE_{011} radio-frequency cavity. of width a, height b and length d, where the field on axis is

$$\begin{cases} B_y(z,t) = -B_0 \frac{b}{d} \cos\left(\frac{\pi z}{d}\right) \cos\omega t \\ E_x(z,t) = -\omega B_0 \frac{b}{\pi} \sin\left(\frac{\pi z}{d}\right) \sin\omega t \end{cases}, \quad (1)$$

Fringe Fields

At the cavity entrance and exit these equations become [1]

$$\text{Entrance} \rightarrow \begin{cases} B_y(0,0) = -B_0 \frac{b}{d} \\ E_x(0,0) = 0 \end{cases}, \quad (2)$$

$$\text{Exit} \rightarrow \begin{cases} B_y(d,\tau_{\rm rf}) = -B_0 \frac{b}{d} \cos\pi \cos 2\pi = B_0 \frac{b}{d} \\ E_x(d,\tau_{\rm rf}) = -\omega \frac{b}{\pi} \sin\pi \sin 2\pi = 0 \end{cases}. \quad (3)$$

Both $B_y(0,0)$ and $B_y(d,\tau_{\rm rf})$ rapidly vanish to zero along a small distance $|\delta|$ just outside the cavity (see Fig.1). Assume

$$\begin{cases} [B_y]_{\rm in} = -B_0 \frac{b}{d}\, g(z) & \text{with } g(-\delta) = 0,\ g(0) = 1 \\ [B_y]_{\rm out} = B_0 \frac{b}{d}\, h(z) & \text{with } h(d) = 1,\ h(d+\delta) = 0 \end{cases}. \quad (4)$$

In this cavity a relativistic fermion, entering with its spin directed along the y-axis will experiences a Stern-Gerlach force parallel to the z-axis:

$$f_z = \mu^* C_{zy} \qquad \text{(from eq. (28) of Ref. [1])}, \quad (5)$$

where (see eq.(30) of Ref. [1]).

$$C_{zy} = \gamma^2 \left[\left(\frac{B_y}{\partial z} + \frac{\beta}{c} \frac{\partial B_y}{\partial t} \right) - \frac{\beta}{c} \left(\frac{\partial E_x}{\partial z} + \frac{\beta}{c} \frac{\partial E_x}{\partial t} \right) \right]. \quad (6)$$

The electric field E_x and its derivatives in eq.(6) are almost constantly zero, because of the boundary conditions on the walls of the cavity and their values at the extreme points. Also, the function $\partial B_y / \partial t$ is almost zero along the fringe segments. Consequently eq. (6) reduces to

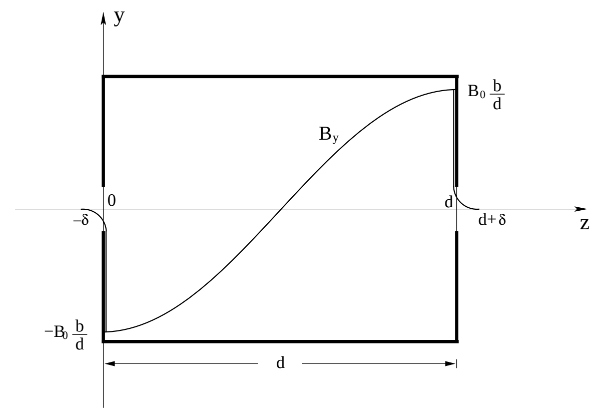

Figure 1: Edge fields at both ends of a single cavity

$$C_{zy} \simeq \gamma^2 \partial B_y / \partial z. \quad (7)$$

Hence we have

$$\begin{cases} [f_z]_{\rm in} = -\mu^* B_0 \frac{b}{d} \gamma^2 \left(\frac{dg(z)}{dz} \right) \\ [f_z]_{\rm out} = \mu^* B_0 \frac{b}{d} \gamma^2 \left(\frac{dh(z)}{dz} \right) \end{cases}. \quad (8)$$

Using eqs. (7) and (8) the energy increments $[\Delta U]_{\rm in}$ and $[\Delta U]_{\rm out}$ related to the fringe fields are easily evaluated since the integrals $\int_{-\delta}^{0} f_z dz$ and $\int_{d}^{d+\delta} f_z dz$ only depend upon the extreme points (4) but not on the connecting curve. $f_z dz$ becomes an exact differential and we obtain

$$[\Delta U]_{in} = [\Delta U]_{out} = -\mu^* B_0 \frac{b}{d} \gamma^2. \quad (9)$$

Total Energy Contribution

A fermion charged particle crossing such radio-frequency $\mathrm{T}E_{011}$ resonator increments its energy by

$$\Delta U = \int_0^d f_z dz = \int_0^d \mu^* C_{zy} dz, \quad (10)$$

that in the ultra relativistic case becomes

$$\Delta U = 2\mu^* B_0 \frac{b}{d} \gamma^2 \qquad \text{(see eq.(54) of Ref. [1]).} \quad (11)$$

Here, summing the fringe (9) to the cavity crossing contribution (11), we obtain

$$[\Delta U]_{tot} = (-1 - 1 + 2)\mu^* B_0 \frac{b}{d} \gamma^2 = 0, \quad (12)$$

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[1] From eqs.(43), (44) of ref.[1] and by choosing $\beta_{\rm ph} = 2\beta$ we obtain $\omega = \frac{2\pi\beta c}{d}$ and $\omega t = \frac{\omega z}{\beta c} = \frac{2\pi\beta c z}{d\beta c} = \frac{2\pi z}{d}$ giving $\omega\tau_{\rm RF} = \frac{2\pi d}{d} = 2\pi$.

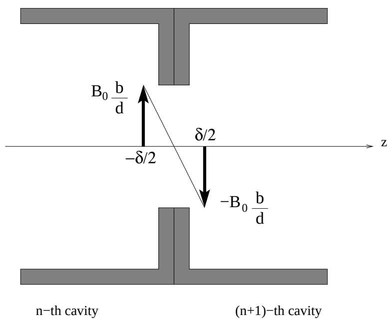

Figure 2: Field gradient between two contiguous cavities

which implies a full cancellation of any energy gain/loss.. The same applies for an array of cavities, since for two contiguous cavities in the middle of the array, there will be a gradient between the positive B_y at the end of the first cavity and a negative B_y at the beginning of the second (Fig.2). Even considering a magnetic field as linearly dependent on z, that is

$$[B_y(z)]_{\rm X} = -2B_0 \frac{b}{d\delta} z, \tag{13}$$

repeating what done before, we will obtain

$$\frac{\partial}{\partial z}[B_y(z)]_{\rm X} = -2B_0 \frac{b}{d\delta} \tag{14}$$

$$f_z = -2\mu^* B_0 \frac{b}{d\delta}\gamma^2 \tag{15}$$

$$\Delta U = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} f_z dz = -2\mu^* B_0 \frac{b}{d}\gamma^2, \tag{16}$$

i.e. for N cavities, we will have as final result

$$[\Delta U]_{tot} = N\Delta U - (N-1)\Delta U - [\Delta U]_{in} - [\Delta U]_{out} = 0. \tag{17}$$

Zero or vanishing small energy variations are also found [2] in the case of traveling waves. Therefore, the conclusion is that a particle with its spin pointing always in the same direction cannot significantly exchange energy with an e.m. wave via the Stern-Gerlach force.

PRECESSING MAGNETIC MOMENT

Let us now examine what happens if the magnetic moment is made to precess. To this aim let's revisit the example [1] of a standing wave in two radio-frequency resonators, tuned in opposite polarity, placed in the gap of a superconducting magnetic dipole with a strong magnetic field $\vec{B}_{\rm M}$ parallel to the x-axis (see Fig.3). This device is inserted in a particle storage ring. To first order assume that the particle trajectory be almost straight. Then, the angle θ subtended by an arc of trajectory is (see Fig.4)

$$\theta \simeq z/\rho. \tag{18}$$

Appropriate spin rotators, installed before and after this pair of cavities, can make in/out spins parallel to the z-axis.

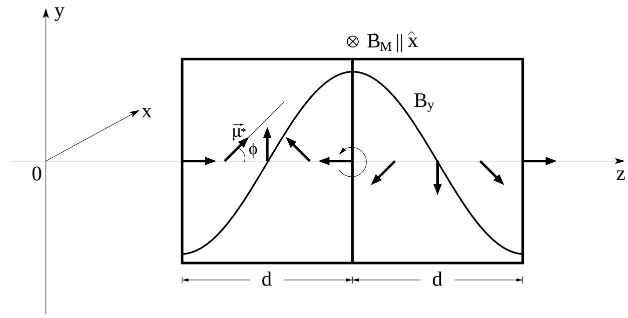

Figure 3: Display of how two π-out-of-phase cavities can yield a non null energy exchange if combined with a 180^0 spin precession here sketched.

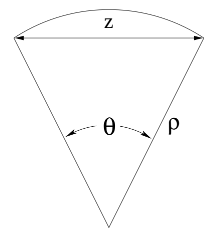

Figure 4: arc of trajectory

The B_y components are made to be contiguous at the center of this setting with opposite gradients at both ends, thus making the edge effects cancel each other. We will show that in these conditions a net energy gain or loss will be produced. The particle's magnetic moment performs a full Larmor rotation in a distance $2d$ after leaving both cavities, and at a generic distance z the spin precession ϕ is

$$\phi = \pi/z. \tag{19}$$

In terms of the precession angle ϕ, the anomaly a, the Lorentz factor γ and the deflection angle θ (related as follows in the Ultra Relativistic limit, UR)

$$\phi = a\gamma\theta \simeq a\gamma\frac{qB_M z}{\beta\gamma mc} = \frac{aqB_M z}{\beta mc} \simeq \frac{aqB_M z}{mc}, \quad (UR) \tag{20}$$

the vertical component of the magnetic moment is

$$\mu_y = \mu^* \sin\phi = \mu^* \sin\left(\frac{aqB_M z}{\beta mc}\right). \tag{21}$$

Comparing eqs. (19) and (20) we obtain

$$B_{\rm M} d = \beta\pi\frac{mc}{aq} \simeq \pi\frac{mc}{aq} \quad (UR), \tag{22}$$

where $d = \frac{1}{2}\beta_{\rm ph}\lambda_{\rm rf}$ by definition. The deflection θ, albeit small, repeats after a huge number (of the order of 10^6) of revolutions and would cause a continuous displacement of the beam, that may be compensated by another magnet, which contains another pair of cavities, with its field anti-parallel to the field of the first. The higher the beam energy, the smaller the deflection. With parameters for a

Table 1: Physical quantities relevant to our spin rotation

Particle	$\frac{mc}{q}$ [Tm]	$\frac{mc}{aq}$ [Tm]	$\pi\frac{mc}{q}$ [Tm]
$e^{\pm}$	1.67×10^{-3}	1.44	4.52
$p\bar{p}$	3.13	1.74	5.46

Table 2: Possible radio-frequency data

Dipole	$B_{\rm M}$ [T]	$d = \frac{1}{2}\beta_{\rm ph}\lambda_{\rm rf}$ [cm]	$\frac{f}{\beta_{\rm ph}}$ [MHz]
LHC	8.33	33	454
NED	15	18	833

reasonable setup given in Table 1, e. g. using data of the existing LHC dipoles [4] and the planned Next European Dipole [5], we find the data of Table 2. The working principle of this proposal seems correct, but has to be further investigated.

The magnetic moment during precession in the (z, y)-plane exhibits a component

$$\mu_z = \gamma\mu^* \cos\phi, \tag{23}$$

which introduces a further term in the longitudinal component f_z of the Stern-Gerlach force whose energy contribution can be neglected [3] since it fades away as $1/\gamma$.

PRELIMINARY ANALYTICAL CONSIDERATIONS

The new feature of this proposal consists in considering the pure Larmor rotation of the particle spin in both frames of references (particle and laboratory), having ignored the Thomas precession because the trajectory can be considered almost rectilinear in the cavity. Therefore the quantity μ^*, which appears in eq.(49) of Ref. [1], will be replaced with its varying version $\mu^* \sin\phi$ of eq. (21). The result is:

$$\begin{aligned} f_z = \quad & \mu^* \sin\phi\gamma^2 B_0\, b\left\{\frac{1}{\pi}\left[\left(\frac{\pi}{d}\right)^2 + \left(\frac{\beta\omega}{c}\right)^2\right]\right. \\ & \left. \times \sin\left(\frac{\pi z}{d}\right)\cos\omega t + \frac{2}{d}\left(\frac{\beta\omega}{c}\right)\cos\left(\frac{\pi z}{d}\right)\sin\omega t\right\}, \end{aligned} \tag{24}$$

and the energy contribution is

$$\delta U = \int_0^{2d} f_z dz = \gamma^2 B_0 \mu^* b\, I, \tag{25}$$

or, after some calculations,

$$\delta U = 2\beta^3\gamma^2 B_0\mu^* \; \frac{b}{d}\; \frac{\beta_{\rm ph}^2 - 1 - \beta^2\beta_{\rm ph}^2}{\beta_{\rm ph}(\beta_{\rm ph}^2 - 4\beta^2)}\; \sin\left(2\pi\frac{\beta_{\rm ph}}{\beta}\right). \tag{26}$$

Recalling equations in section 4 of Ref. [1], such as

$$\beta_{\rm ph} = \sqrt{1 + \left(\frac{d}{b}\right)^2} \quad\rightarrow\quad \frac{b}{d} = \frac{1}{\sqrt{\beta_{\rm ph}^2 - 1}} \tag{27}$$

eq. (26) becomes

$$\begin{aligned} \delta U = \quad & -2\beta^3\gamma^2 B_0\mu^* \left[\frac{1}{\sqrt{\beta_{\rm ph}^2-1}}\; \frac{1-\beta_{\rm ph}^2\gamma^{-2}}{\beta_{\rm ph}(\beta_{\rm ph}^2-4\beta^2)}\right] \\ & \times \sin\left(2\pi\frac{\beta_{\rm ph}}{\beta}\right), \end{aligned} \tag{28}$$

which for ultra-relativistic particles reduces to

$$\delta U \simeq -\gamma^2 B_0\mu^* \; \frac{2}{\beta_{\rm ph}\sqrt{\beta_{\rm ph}^2 - 1}\,(\beta_{\rm ph}^2 - 4)} \; \sin(2\pi\beta_{\rm ph}), \tag{29}$$

or

$$\delta U = -0.91\,\gamma^2 B_0\mu^*. \tag{30}$$

For $\beta_{\rm ph} = 2$ the radio-frequencies to be employed are

$$f_{\rm rf} = \left\{ \begin{array}{ll} 908\ {\rm M}Hz & (B_{\rm M} = 8.33\ {\rm T}) \\ 1.02\ {\rm G}Hz & (B_{\rm M} = 15\ {\rm T}) \end{array} \right. . \tag{31}$$

In the former study [1] of the Stern-Gerlach interaction, μ_y was set equal to μ^* (constant) and, after crossing a single radio-frequency resonator and neglecting the edge effects, for $\beta_{\rm ph} = 2$ we obtained $\delta U = 2\frac{b}{d}\gamma^2 B_0\mu^* = 1.15\ \gamma^2 B_0\mu^*$. In the present case, with two cavities crossed, we have found the factor 0.91 shown in eq. (30) which is 40% smaller than $2 \times 1.15 = 2.30$. This decrease is due to the use of $\mu_y = \mu^* \sin\phi$, *i.e.* of a quantity whose modulus varies continuously from 0 to μ^*.

Another solution could consist in setting $\sin(2\pi\beta_{\rm ph}) = \pm 1$ by choosing $\beta_{\rm ph}$ equal to an odd number divided by four. A glance at the factor $2/\left[\beta_{ph}\sqrt{\beta_{ph}^2 - 1}(\beta_{ph}^2 - 4)\right]$, appearing in eq. (29), suggests to set $\beta_{\rm ph} = \frac{5}{4} = 1.25$ which results in

$$\delta U = -0.87\,\gamma^2 B_0\mu^*. \tag{32}$$

CONCLUSIONS

At this stage of the game we plan to revisit our previous work on the relativistic time varying Stern-Gerlach interaction. In particular, we would like to start with the experimental verification of the γ^2-law, postponing the study of a spin splitting device. The main issue of this paper is the demonstration of how the cavity fringe fields may cancel any energy variations, an effect that can be overcome using spin precession.

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