LARGE DISPLACEMENT AND DIVERGENCE ANALYTIC TRANSFER MAPS THROUGH QUADRUPOLES

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Abstract

Linear-field non-scaling FFAGs are proposed for multi-GeV acceleration of muons and order hundreds MeV/u proton or carbon for medical applications. The large momentum acceptance lattices employ alternating focusing and defocusing combined-function magnets. In one implementation, rectangular quadrupole magnets are used, with the dipole component generated by off-setting the magnet centre relative to the reference orbit. This feature, coupled with the large radial aperture for momentum variation, gives rise to large amplitude orbits. The angles are so large that there is a partial interchange of the longitudinal and transverse momentum relative to the fixed coordinate system of the quadrupole. We examine methods to devise non-linear transfer maps to high order. The map is constrained by the fact that its first partial derivative produces the linear transfer matrix which must have unity determinant.

INTRODUCTION

A rectangular combined function magnet is essentialy a displaced quadrupole. If the magnet achieves a large bending, then the displacements from the magnetic centre are large. Thus a correct treatment demands consideration of large amplitude oscillations. Treatments of these third-order aberrations are given in Refs.[1, 2]. Our development differs in two respects: we pay close attention to conservation of the unity determinant - which property is not guaranteed in [1, 2] since these works are inspired by single-pass through a spectrometer; we retain time as the independent variable rather than distance along the orbit.

Initially, we had considered the WKBJ approximation as a method for dealing with this problem. A quadrupole with varying longitudinal velocity "looks" like a quadrupole with varying focusing strength k, but there are problems. the $1^{\rm st}$-order WKBJ technique leads to a vaying determinant; this is "fixed" in the $2^{\rm nd}$-order WKBJ treatment but leads to complicated matrix elements. Another problem is that elements contain trig' functions with argument $k\int_0^t dt\sqrt{1+v_z(t)}$ where v_z is the incremental longitudinal velocity. If the square root is not expanded, then numerical integration must be used; and the low order expansion breaks down in the parameter regime in which the corrections are important. Compared with that delicate situation, the Green's function[3] of the simple, linear quadrupole provides an accurate and robust method of computing trajectories.

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EQUATIONS OF MOTION AND THEIR SOLUTION

We shall study bending in the horizontal plane, We adopt a cartesian system of particle coordinates $\mathbf{x} = [x(t), y(t), v_s t + z(t)]$ with x, y horizontal and vertical, respectively, and z aligned with the symmetry axis of the quadrupole. The particle charge, rest mass and speed are e, m_0 and v_s. The quadrupole gradient and strength are B_1 Tesla/m $k^2 = (eB_1)/(\gamma v_s m_0)$. We introduce the divergences$(d/dt)[x,z] = v_s(d/ds)[x,z] = v_s[x',z']$, where $s = v_s t$. The equations of motion are:

$$\begin{aligned} (dx'/ds) &+ k^2 x(1+z') = 0 && (1)\\ (dy'/ds) &- k^2 y(1+z') = 0 && (2)\\ (dz'/ds) &+ k^2(yy' - xx') = 0\ . && (3) \end{aligned}$$

The initial conditions at $s = 0$ are

$$\mathbf{x}_0 = [x_0, x_0', z_0, z_0', y_0, y_0']\ . \qquad (4)$$

The last equation (3) can be integrated immediately. In the case $y^2 \ll x^2$, this becomes

$$z'(s) = z_0' + (k^2/2)[x^2(s) - x_0^2]\ . \qquad (5)$$

For motion confined to the horizontal plane the divergences satisfy

$$(1+z')^2 + (x')^2 = 1\ . \qquad (6)$$

1st & 2nd Approximation

The first step is to treat the terms in z' as negligible, leading to SHM with the well-known solution

$$[x(s), x'(s)] = \mathbf{T}_0[x_0, x_0'] \ \text{ with } \ \mathbf{T}_0 = \left[\begin{array}{cc} C & S \\ C' & S' \end{array}\right]. \qquad (7)$$

Here $C(s), S(s)$ are the principal functions. There is an analogous solution for y, y' which we shall continue to omit for brevity. We restore the perturbation W_x :

$$(dx'/ds) + k^2 x = -k^2 x[z_0' + (k^2/2)(x^2 - x_0^2)] \equiv W_x(s) \qquad (8)$$

We treat this as if it were an inhomogeneous equation and solve by the method of Green's functions. Substituting (7), the perturbations are

$$\begin{aligned} W_x &= -k^2(C_x x_0 + S_x x_0')z'(s) && (9)\\ z'(s) &= z_0' + (k^2/2)[(C_x x_0 + S_x x_0')^2 - x_0^2] && (10)\\ z(s) &= z_0 + \int_0^s z'(u)du\ . && (11) \end{aligned}$$

The solution is:

$$x(s) = C_x x_0 + S_x x_0' + \int_0^s G_x(s,u) W_x(u) du \quad (12)$$

$$x'(s) = C_x' x_0 + S_x' x_0' + \int_0^s G_x'(s,u) W_x(u) du \quad (13)$$

Our working is not yet complete, in the next section we show the need for introducing a third approximation. Explicit expressions for (x, x') are given in Ref. [4].

Transfer matrix for small oscillations

The above exercise will find any number of trajectories; and we may single out one of them as *the* reference trajectory. Then we may ask what is the small amplitude motion about the reference. The equations (10-13) constitute a set of nonlinear mappings which take the old coordinates into new coordinates at later times. Symbolically $x(s) = T_x(s, \mathbf{x}_0)$ and $x'(s) = T_{x'}(s, \mathbf{x}_0) = (\partial/\partial s)T_x$, etc. For brevity we adopt $\mathbf{x}(s) = [x_1, x_2, x_3, x_4, x_5, x_6] = [x, x', z, z', y, y']$, so the mapping is written $x_i = T_i(s, \mathbf{x}_0)$

If we treat the particular set of initial conditions $\mathbf{x}_0$ as defining a reference trajectory, then the transport of small deviations is given by the matrix of partial derivatives $T_{ij} = \partial T_i(s, \mathbf{x}_0)/\partial x_{j,0}$.

The underlying equations are conservative because only magnetic fields are present; and so the determinant of the transfer matrix, $\mathrm{Det}[T_{ij}]$, should be unity. In fact, our expressions for the elements T_{ij} are only approximate, and $\mathrm{Det}[T_{ij}]$ will deviate from unity.

$$\mathrm{Det}[T_{ij}] = 1 + k^4[x_0^2 s^2/2 + x_0 x_0' s^3(2/3) + \ldots] . \quad (14)$$

The error would be appreciably reduced if the quadratic and cubic error terms (s^2, s^3) could be eliminated.

3rd Approximation

In large part the error in the determinant arises because the longitudinal motion $[z, z']$ is a lower order approximation than the transverse $[x, x']$.

The third approximation is to substitute the perturbed x, x', equations (12,13), into z', expression (5). The resulting expression for z' contains powers of x_0, x_0' up to six, which is an unncessarily high order. Consequently, we eliminate all terms $(x_0)^n$ and $(x_0')^n$ with $n > 3$, but retain $(x_0)^2(x_0')^2$. Explicit expressions for (z, z') are given in Ref. [4]. The determinant is

$$\begin{aligned} \mathrm{Det}[T_{ij}] &= 1 + (1/12)k^4 s^4 (k^2 x_0^2 + z_0')^2 \\ &+ (1/4)k^6 s^5 (k^2 x_0^2 + z_0') x_0 x_0' + \ldots \end{aligned} \quad (15)$$

The identity (6) implies that we may substitute $-(x_0')^2/2$ in place of z_0' (to 1$^{\mathrm{st}}$ order). For example, if the total bending produced by the quadrupole is 2θ, the entry and exit angles are each $\pm\theta$ and the error (compared with unity) is $(ks\theta)^4/48$. Now to achieve a bend angle 2θ requires $k^2 s^2 \approx 2\theta$, and so the error is $(\theta)^6/12$.

EXAMPLE

Consider the following numerical example: a 100 MeV kinetic energy proton beam, momentum 444.6 MeV/c, a field gradient $B_1 = 5$ T/m, and quadrupole of length 0.8555 m. The k-value is 1.836 per metre. We shall diagram trajectories for four sets of initial conditions A-D.

Case	x_0 (m)	x_0' (radian)	bend angle	Figs.
A	0.5	0	$54°$	1,2
B	0	0.5	$29°$	3,4
C	0.25	0.5	$56°$	5,6
D	0.25	-0.5	$-1°$	7,8

Relative fractional Errors

We shall plot the relative errors for trajectories calculated with and without the order $x_0^3, (x_0')^3$ corrections computed via the Green's functions. There is a significant increase of the accuracy of x, x' as compared to the simple formulae for a quadrupole with constant focusing strength. We show also the relative errors in z, z' when the longitudinal motion is taken to order $x_0^3, (x_0')^3$ (Green's function) compared with order $x_0^2, (x_0')^2$ (simple, linear quadrupole); the increase in accuracy is not so great, but is essential to reducing errors in the determinant.

Throughout, the figures are colour-coded. Green's function: x (red) and x' (blue) Green's function z (red) and z' (blue). Simple, linear quad x (magenta) and x' (cyan). Simple, linear quadrupole z (magenta) and z' (cyan).

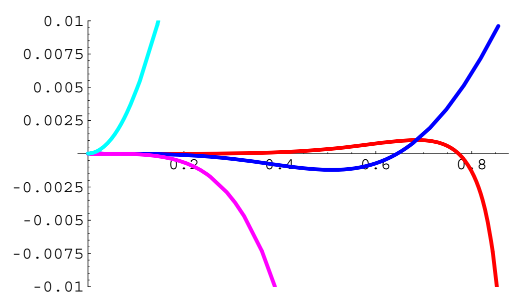

Figure 1: Case A. x, x'

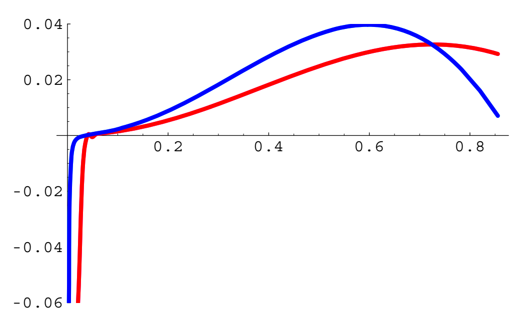

Figure 2: Case A. z, z'

In this particular case, A, the aberrations in z, z' are so small that the Green's function and simple linear quadrupole formulae produce indistinguishable results.

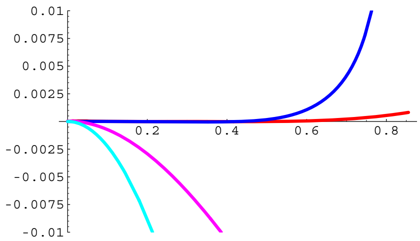

Figure 3: Case B. x, x'

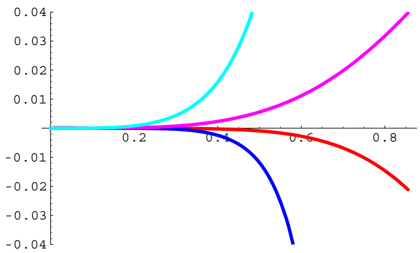

Figure 4: Case B. z, z'

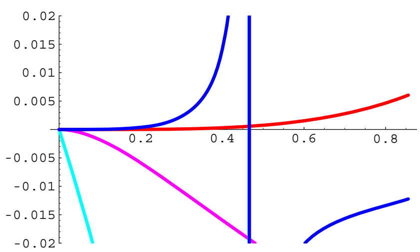

Figure 5: Case C. x, x'

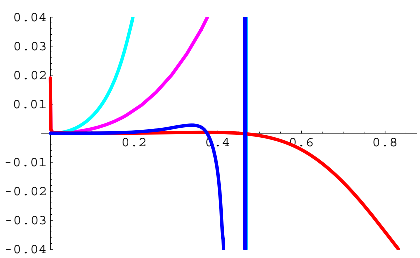

Figure 6: Case C. z, z'

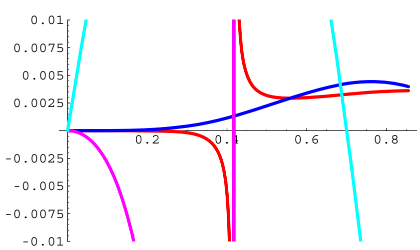

Figure 7: Case D. x, x'

Determinant

Figures 9,10 show the evolution of the determinant through the quadrupole. The formula (15) explains quite well the relative size of the errors in Fig. 10; e.g. partial cancellation between s^4 and s^5 terms in case D accounts for the error. The other important point is that the high power law for the error $(s^4 + \ldots)$ implies that enormous gains are made in accuracy by simply subdividing the range.

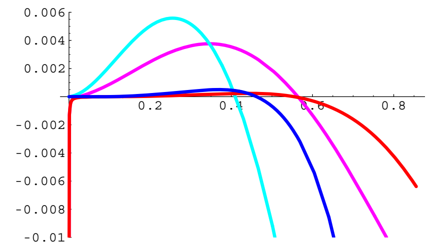

Figure 8: Case D. z, z'

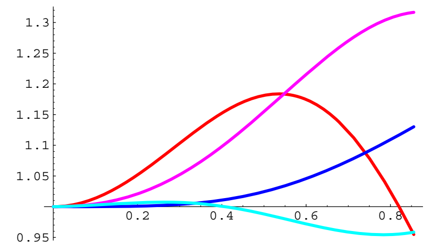

Figure 9: Variation of determinant before correction. Case A (red), B (blue), C (magenta), D (cyan).

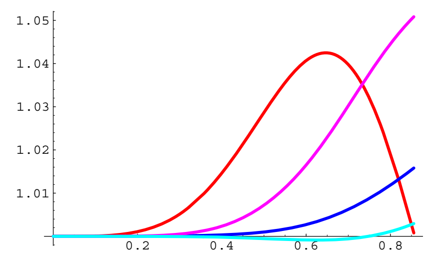

Figure 10: Variation of determinant after correction.

CONCLUSION

We have presented formulae for large amplitude and angle motion in a rectangular quadrupole magnet. Contrary to the case of a spectrometer, where the particle beam makes a single passage, we are concerned with periodic traversals of a circular lattice; and conservation of the determinant is important. By taking the longitudinal motion to higher order, we have reduced the error in the matrix determinant for small oscillations from order s^2 to order s^4 in the longitudinal coordinate. For further details, see Ref.[4].

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