EQUILIBRIUM BEAM DISTRIBUTION IN ELECTRON STORAGE RINGS NEAR SYNCHROBETATRON COUPLING RESONANCES

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Abstract

Based on the work [1],[2], we discuss the topic of finding the equilibrium beam distribution near synchrobetatron coupling resonances. We show how this framework reduces to the Sands results in the uncoupled case. A variety of damping/diffusion effects can be included. We discuss the case of IBS damping diffusion in detail, showing how it can be combined with our analytical treatment of emittance evolution. For the synchrobetatron coupled case, this formalism has been applied in detail to the case of dispersion at RF cavities and crab cavities.

INTRODUCTION

The beam distribution in an electron storage ring can often be descibed by a Gaussian function of phase space. In the case where the dynamics is approximately linear, then motion along the 3 invariants dominates and the distribution can further be described in the form

$$f(\vec{z}) = \frac{1}{\pi^3\langle g_1\rangle\langle g_2\rangle\langle g_3\rangle}\exp\left(-\frac{g_1}{\langle g_1\rangle}-\frac{g_2}{\langle g_2\rangle}-\frac{g_3}{\langle g_3\rangle}\right). \tag{1}$$

where the g_a ($a = 1, 2, 3$) are the eigeninvariants (action variables) of the linear dynamics. They are quadratic quantities in the phase space variables z_i: $g_a = \vec{z}^T G_a \vec{z}$ and can be computed from the eigenvectors of the one turn map. The average values of these quantities, the $\langle g_a\rangle$ are related to the rms emittances ϵ_a by $\langle g_a\rangle = 2\epsilon_a$. Under these conditions, then, the problem of finding the beam distribution reduces to finding the form of the invariants, g_a, and then to finding the evolution of, or equilibrium values of the emittances. In particular, assuming that the one turn map is not changing, the problem of beam distribution reduces to finding the evolution of three quantities, even in a strongly coupled case.

There are thus two pieces to describing the beam distribution: first, finding the invariants, and second, finding the evolution of the emittances. Regarding the first part, we formulate a perturbation theory to find approximate expressions for the invariants, particularly near linear resonances. For the second part, damping and diffusion effects will change the beam moments which then affects the beam emittances. This can be captured by the equation

$$\Delta\langle g_a\rangle = \oint ds\, \mathrm{Tr}\left[G_a \frac{d\Sigma}{ds}\right] \tag{2}$$

with Tr representing the trace of a matrix and the integration being around the ring to find the total change. Here, $\Sigma_{ij} = \langle z_i z_j\rangle$ is second moment matrix for the distribution. In case the damping and diffusion have no phase space dependence, the evolution equation takes on the simple form

$$\Delta\langle g_a\rangle = -2\chi_a\langle g_a\rangle + \bar{d}_a \tag{3}$$

where χ_a are global damping decrements and $\bar{d}_a$ global diffusion coefficients. Thus, we immediately write down the equilibrium solution

$$\langle g_a\rangle_{\mathrm{eq}} = \frac{\bar{d}_a}{2\chi_a}, \qquad a = 1, 2, 3. \tag{4}$$

In the case when the diffusion and damping have phase space dependence, one must go back to Eqn. (2). In this way, the emittance effects of physical phenomenon such as intrabeam scattering, or beam-beam diffusion can be computed.

PERTURBATION THEORY CALCULATION OF INVARIANTS

As stated in the introduction, the first part of finding the equilibrium beam distribution involves finding the invariants of the one-turn map matrix. And these invariants can be constructed from the eigenvectors. Previous perturbative treatments that cover resonances (e.g. [10]) have focussed directly on the invariants (actions – i.e. canonical perturbation theory!), requiring an additional two invariants near resonance. Ref. [17] uses a similar perturbative approach, but does not include the case of resonance. When formulated in terms of the eigenvectors, no additional quantities are needed. In addition to this, the connection to the Quantum Mechanical eigenvalue problem, clarifies this approach.

The main object of study is the one turn map matrix M. M is symplectic [3] which means

$$M^T J M = J, \tag{5}$$

where a superscript T means taking the transpose of a matrix, and J is the symplectic inner product matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{6}$$

The six eigenvectors and eigenvalues of M satisfy

$$M v_k = \lambda_k v_k. \tag{7}$$

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The index k runs over $\pm1, \pm2, \pm3$. We normalize the eigenvectors such that

$$v_j^\dagger J v_k = i\,\mathrm{sgn}(j)\,\delta_{jk}, \tag{8}$$

where $\mathrm{sgn}(j)$ is 1 for $j > 0$ and -1 for $j < 0$, and † means taking the complex conjugate and transpose of a matrix (or vector). This normalization condition suggests the definition of an upper indexed object

$$v^j \equiv -i\,\mathrm{sgn}(j)\,v_j^\dagger J. \tag{9}$$

The normalization condition (8) then reads

$$v^j v_k = \delta_{jk}. \tag{10}$$

The invariant matrices are given from these eigenvectors via

$$G_a = -J(v_a v_a^\dagger + v_a^* v_a^T)J. \tag{11}$$

with * representing complex conjugate, † representing conjugate transpose and J the symplectic inner product matrix given in Eq. (6).

We start with the uncoupled case. We consider motion in the transverse-longitudinal ($x - z$) plane.

Uncoupled Ring

A typical storage ring is designed to be uncoupled. If we use betatron coordinates, defined by $\vec{z}_\beta = \mathcal{B}\vec{z}$ with $\mathcal{B}$ a dispersion matrix,

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & -\eta \\ 0 & 1 & 0 & -\eta' \\ \eta' & -\eta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{12}$$

with η the dispersion and $\eta' = d\eta/ds$. Then the linear one-turn map for the ring at some position s is of the form

$$M_{\mathrm{uncoupled}} = \begin{pmatrix} M_x & 0 \\ 0 & M_z \end{pmatrix}. \tag{13}$$

Because M_x and M_z are symplectic, following Courant and Snyder, we can write them in the form

$$M_x = \cos\mu_x I + \sin\mu_x J_x = e^{\mu_x J_x}, J_x = \begin{pmatrix} \alpha_x & \beta_x \\ -\gamma_x & -\alpha_x \end{pmatrix}, \tag{14}$$

and

$$M_z = \cos\mu_z I + \sin\mu_z J_z = e^{\mu_z J_z}, J_z = \begin{pmatrix} \alpha_z & \beta_z \\ -\gamma_z & -\alpha_z \end{pmatrix}. \tag{15}$$

Here, β_x, $\alpha_x = -2\beta_x'$, and $\gamma_x = \frac{1+\alpha_x^2}{\beta_x}$ are the usual Courant-Snyder lattice parameters. They are periodic with period C; *e.g.* $\beta_x(s+C) = \beta_x(s)$. Note that adding integer multiples of 2π to μ_x and μ_z does not change the one turn map. We will thus, except where otherwise noted, assume that an appropriate multiple has been added (subtracted) so that

$$\mu_{x,z} \in [-\pi, \pi] \tag{16}$$

We can also find corresponding lattice parameters for z. For the case of a single RF cavity, and taking lowest order in μ_z we can show

$$\beta_z = \frac{a}{\mu_s}, \quad \gamma_z = \frac{\mu_s}{a}, \quad \alpha_z = \frac{-\mu_s}{2}(1 - 2\breve{\alpha}) \tag{17}$$

where $a = C\alpha_c$, with C the ring circumference and α_c the momentum compaction factor. We have taken the case of above transition in which $\mu_z = -\mu_s$. Also,

$$\breve{\alpha} = \frac{1}{a}\int_{s_c}^{s} \frac{\eta(s')}{\rho(s')} ds', \tag{18}$$

may be thought of as the partial momentum compaction factor between the cavity and the point of observation.

Given the general forms (14) and (15) for M_x and M_z, we can express the eigenvectors of $M_{\mathrm{uncoupled}}$ as

$$v_x = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\beta_x} \\ \frac{i-\alpha_x}{\sqrt{\beta_x}} \\ 0 \\ 0 \end{pmatrix}, \qquad v_z = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ \sqrt{\beta_z} \\ \frac{i-\alpha_z}{\sqrt{\beta_z}} \end{pmatrix}, \tag{19}$$

with corresponding eigenvalues $e^{i\mu_x}$ and $e^{i\mu_z}$, which can be checked by direct multiplication.The above v_x and v_z are positive modes, and the corresponding negative modes are $v_{-x} = iv_x^*$ and $v_{-z} = -iv_z^*$. Using the notation in (9), we can express the normalization as $v^x v_x = v^z v_z = v^{-x} v_{-x} = v^{-z} v_{-z} = 1$ and all other combinations give 0. To be explicit, because v_x is a positive mode (likewise v_z), $v^x = -iv_x^\dagger J$.

Adding a perturbation

We would like to be able to include the case of resonances in our perturbation theory. This will require an additional step beyond perturbing the uncoupled one-turn map. In particular, we must start with the one-turn map *exactly* on resonance and consider both coupling and difference from resonance as perturbations. Explicitly, we write

$$M = M_0 + M_{1\mu} + M_{1\xi} \tag{20}$$

where $M_{1\mu}$ is the difference from resonance and $M_{1\xi}$ is a coupling perturbation. We consider both of these matrices to be first order in some small parameter ϵ. This will suffice for sum and difference resonances, but it turns out that this needs to be reexamined to correctly treat integer and half-integer resonances. See [1] and [2] for more details.

These ideas, along with the realization that we are dealing with a symplectic operator, rather than a Hermitian one, allow us to formulate the perturbation theory in a quite clear manner.

To discuss the results of the perturbation analysis, let us first write the perturbation in the form

$$M_1 = PM_0 \tag{21}$$

Now, consider the quantities.

$$r_{mn} = v^{m0} P v_{n0}, \tag{22}$$

The v_{n0} are the eigenvectors of M_0. These are analogues of the matrix elements in quantum mechanical perturbation theory. The only difference here is that since the underlying operator is symplectic instead of Hermitian, the symmetry properties of these quantities are different. [1] We find that all our results can be expressed in terms of the r_{mn}. In the case of a sum and difference resonance, we find that the coupled eigenvectors involve a coupling angle θ. For the difference resonance with a purely coupling perturbation P,

$$\tan\theta = \frac{2|r_{12}|}{\mu_x - \mu_z} \tag{23}$$

and for the sum resonance,

$$\tanh\theta = \frac{2|r_{1-2}|}{\mu_x + \mu_z} \tag{24}$$

Crab Cavity

We consider a single crab cavity[18]. The map for the crab cavity is given by

$$T_{\mathrm{crab}} = I + P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \xi_c & 0 \\ 0 & 0 & 1 & 0 \\ \xi_c & 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

where ξ_c gives the strength of the cavity. In the case where the crab cavity is used to correct for a half crossing angle Φ at the interaction point of a collider [7], assuming no resonance, ξ_c is related to that crossing angle by

$$\xi_c = \begin{cases} \frac{2\Phi\sin(\pi\nu_x)}{\sqrt{\beta_k\beta_x^*}} & \text{single crab cavity} \\ \frac{\Phi}{\sqrt{\beta_k\beta_x^*}} & \text{crab cavity pair} \end{cases} \tag{26}$$

where β_k is the beta function at the crab cavity and β_x^* is the beta function at the interaction point.

Applying our formalism to the perturbation P in (25), we find for the coupling parameter

$$\xi_\pm \equiv 2|r_{\mp12}| = \xi_c\sqrt{\frac{a\beta_x}{\mu_s} \mp 2\eta^2} \tag{27}$$

From this expression, we can immediately see that for small synchrotron tune and near a sum or difference resonance, a crab cavity will cause a large coupling perturbation[2].

INCLUSION OF DAMPING/DIFFUSION TO FIND EMITTANCES

For the case of damping and diffusion due to radiation, the damping matrix is given by

$$B_\beta(s) = \mathcal{B}B\mathcal{B}^{-1} \tag{28}$$

$$= \begin{pmatrix} -\eta b_{\delta x} & 0 & 0 & -\eta(b_z + \eta b_{\delta x}) \\ -\eta' b_{\delta x} & b_x & 0 & \eta'(b_x - b_z - \eta b_{\delta x}) \\ 0 & -b_x\eta & 0 & -b_x\eta\eta' \\ b_{\delta x} & 0 & 0 & b_z + \eta b_{\delta x} \end{pmatrix}.$$

where b_x is the damping effect from the RF cavity, and b_z and $b_{\delta x}$ are the damping effects from bending magnets. The diffusion matrix is given by

$$D_\beta = \mathcal{B}D\mathcal{B}^T = d\begin{pmatrix} \eta^2 & \eta\eta' & 0 & -\eta \\ \eta\eta' & \eta'^2 & 0 & -\eta' \\ 0 & 0 & 0 & 0 \\ -\eta & -\eta' & 0 & 1 \end{pmatrix} \tag{29}$$

with

$$d(s) = \frac{55}{48\sqrt{3}}\alpha_0\frac{\gamma^5}{|\rho(s)|^3}\left(\frac{\hbar}{mc}\right)^2, \tag{30}$$

with α_0 the fine structure constant, γ the relativistic energy factor, $\rho(s)$ the bending radius of the bending magnets, $\hbar$ the reduced Planck constant,and m the electron rest mass. As we see from the $\rho(s)$ in the denominator, diffusion only happens in the bending magnets.

Now that we have the global diffusion coefficients and damping decrements for the uncoupled case, Eq. (4) gives the equilibrium values of the invariants, or in terms of the emittances $\epsilon_a = \langle g_a\rangle/2$ we find

$$\epsilon_x = \frac{\frac{55}{48\sqrt{3}}\alpha_0\gamma^5\oint ds\frac{\mathcal{H}_x}{|\rho^3|}}{\frac{2U_0}{E_0}\mathcal{J}_x} \tag{31}$$

$$\epsilon_z = \frac{\frac{55}{48\sqrt{3}}\alpha_0\gamma^5\frac{a}{\mu_s}\oint ds\frac{1}{|\rho^3|}}{\frac{2U_0}{E_0}\mathcal{J}_z} \tag{32}$$

These are the famous results of Sands [4].

THEORETICAL PROPERTIES OF FRAMEWORK

Here we consider two interesting properties of the framework: sum rules and anti-damping.

Sum Rules

The sum of the local damping coefficients gives the trace of $A(s)$. From this we can derive a global sum rule,

$$\chi_1 + \chi_2 + \chi_3 = \chi_x + \chi_y + \chi_z \tag{33}$$

where $\chi_{x,y,z}$ are the global damping decrements for the uncoupled case. Equation (33) is a manifestation of the well-known Robinson theorem [15].Likewise, under some specific conditions, one can also obtain a sum rule for the invariants and diffusion coefficients. For the invariants, to lowest order, we find:

$$\begin{aligned} G_1 - G_2 &= G_x - G_z \quad \text{sum res.} \\ G_1 + G_2 &= G_x + G_z \quad \text{dif. res.} \end{aligned} \tag{34}$$

and likewise, for the global diffusion coefficients,

$$\bar{d}_1 - \bar{d}_2 = \bar{d}_x - \bar{d}_z = \text{invariant} \quad \text{sum res.}$$

[1] In particular, they satisfy

$$r_{mn} = -\mathrm{sgn}(m)\mathrm{sgn}(n)r_{nm}^*,$$
$$r_{mn} = r_{-m-n}^*.$$

[2] This result is derived in a different way in [5]

$$\begin{array}{rclll} \bar{d}_1+\bar{d}_2 & = & \bar{d}_x+\bar{d}_z = \text{invariant} & & \text{dif. res.} \\ \bar{d}_1 & = & \bar{d}_x & \text{int } z \text{ res.} & \\ \bar{d}_2 & = & \bar{d}_z & \text{int. } x \text{ res.} & \end{array} \tag{35}$$

From the sum rule (35) for sum and difference resonances (global version), and using (4), we have

$$\begin{array}{ll} \chi_1\langle g_1\rangle_{\rm eq}-\chi_2\langle g_2\rangle_{\rm eq}=\text{invariant} & \text{sum res.} \\ \chi_1\langle g_1\rangle_{\rm eq}+\chi_2\langle g_2\rangle_{\rm eq}=\text{invariant} & \text{dif. res.} \end{array} \tag{36}$$

Since $\chi_{1,2}$ must be positive for stable motion, it follows that this sum rule imposes a stability condition for particle motion. For example, in case of coupling between the two transverse betatron motions, Eq.(36) implies that the motion is stable near a difference resonance and possibly unstable near a sum resonance. This is a familiar result involving the sum rule of equilibrium beam emittances [3]. The present formalism therefore contains in one framework the Robinson sum rule and the emittance sum rule near linear resonances.

In the case where the coupling near a sum/difference resonance occurs in synchrobetatron space, and the operation is above transition, we find that $\mu_z=-\mu_s$ (see later) where μ_s is the usual (positive) synchrotron phase advance per turn. Thus, in terms of μ_s, there is a sign reversal in the definition of degeneracy so that a sum resonance has $\mu_{x0}=\mu_s$ and difference resonance has $\mu_{x0}=-\mu_s$. In terms of μ_s, then, stability applies near a sum resonance and instability occurs near a difference resonance. This is also a familiar result [16], associated with the longitudinal negative mass above transition. In the present paper, however, we make the choice to relate our definitions of resonance to ν_z so that as in the case of x-y coupling, the difference resonance is stable and the sum resonance is unstable. This has the advantage of permitting a uniform treatment of synchrobetatron coupling and transverse betatron coupling. To reiterate, by "sum resonance", we mean $\nu_x+\nu_z$ is near an integer, and for a "difference resonance", $\nu_x-\nu_z$ is near an integer.

Emittance Coupling

For the sum resonance, we find

$$\epsilon_1^+ = \frac{\cosh^2\frac{\theta}{2}\bar{d}_x+\sinh^2\frac{\theta}{2}\bar{d}_z}{4(\cosh^2\frac{\theta}{2}\chi_x-\sinh^2\frac{\theta}{2}\chi_z)} \tag{37}$$

$$\epsilon_2^+ = \frac{\sinh^2\frac{\theta}{2}\bar{d}_x+\cosh^2\frac{\theta}{2}\bar{d}_z}{4(-\sinh^2\frac{\theta}{2}\chi_x+\cosh^2\frac{\theta}{2}\chi_z)} \tag{38}$$

while for the difference resonance, we find

$$\epsilon_1^- = \frac{\cos^2\frac{\theta}{2}\bar{d}_x+\sin^2\frac{\theta}{2}\bar{d}_z}{4(\cos^2\frac{\theta}{2}\chi_x+\sin^2\frac{\theta}{2}\chi_z)} \tag{39}$$

$$\epsilon_2^- = \frac{\sin^2\frac{\theta}{2}\bar{d}_x+\cos^2\frac{\theta}{2}\bar{d}_z}{4(\sin^2\frac{\theta}{2}\chi_x+\cos^2\frac{\theta}{2}\chi_z)} \tag{40}$$

Note that in the case where $\chi_x=\chi_z$, we find that

$$\epsilon_1^+ = \cosh^2\frac{\theta}{2}\epsilon_x+\sinh^2\frac{\theta}{2}\epsilon_z \tag{41}$$

$$\epsilon_2^+ = \sinh^2\frac{\theta}{2}\epsilon_x+\cosh^2\frac{\theta}{2}\epsilon_z \tag{42}$$

for the sum resonance and

$$\epsilon_1^- = \cos^2\frac{\theta}{2}\epsilon_x+\sin^2\frac{\theta}{2}\epsilon_z \tag{43}$$

$$\epsilon_2^- = \sin^2\frac{\theta}{2}\epsilon_x+\cos^2\frac{\theta}{2}\epsilon_z \tag{44}$$

for the difference resonance. Thus, in this case, it makes sense to talk about emittance coupling: the effect of the coupling is simply to mix together the equilibrium emittances. If we were talking about transverse $x-y$ coupling, $\chi_x=\chi_z$ would indeed be approximately correct in many situations and this gives a justification for using that concept for betatron coupling. For the case here of synchrobetatron coupling, typically $\chi_x\approx\chi_z/2$, and thus the concept of emittance coupling is not precise.

Anti-damping Instability

The damping decrements for the sum resonance show an interesting effect. One of $\chi_{1,2}$ will become negative for a finite value of θ. Specifically, suppose that $\chi_z>\chi_x$ which is typically the case. Then χ_1 vanishes when

$$\sqrt{\frac{\chi_z}{\chi_x}}=\coth(\frac{\theta}{2}). \tag{45}$$

For θ larger than this, χ_1 becomes negative, and there is an instability. This is analogous to the case where the damping partition number $\mathcal{D}$ is greater than 1, in which case, χ_x is likewise negative, indicating an instability. We refer to this as an "anti-damping instability".

NON-UNIFORM DIFFUSION/DAMPING

When the damping and diffusion effects have a dependence on the phase space, or the phase space distribution, the evolution equations for the emittances become more involved. Intrabeam scattering is an example of this: the damping and diffusion effects depend both on the phase space position and on the distribution itself. The effect of beam-beam interactions can also be included within this framework [19].

Intrabeam Scattering

To find the emittance growth due to IBS, we must first find the growth in the moments due to IBS and then project this onto the invariants (Eq.2). In [6] expressions of the form

$$\frac{d\bar{\Sigma}_{ab}}{dt}=\mathcal{A}\mathbb{K}_{ab} \tag{46}$$

were derived, with

$$\mathcal{A}=\frac{Nr_0^2c}{32\pi^3\beta^3\gamma^4\epsilon_x\epsilon_y\sigma_s\sigma_\delta} \tag{47}$$

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and

$$\mathbb{K}_{ab} = \int d\Omega \frac{h_{ab}}{h_3} \ln h_1 \tag{48}$$

with

$$h_{ab} = (\hat{r}_a \hat{r}_b - \hat{\Delta}_a \hat{\Delta}_b) \tag{49}$$

where $\hat{r}_a$ and $\hat{\Delta}_a$ are perpendicular unit vectors and

$$h_1 = \bar{\mathbb{A}}_{ab} \hat{b}_a \hat{b}_b \tag{50}$$
$$h_3 = \bar{\mathbb{C}}_{ab} \hat{\Delta}_a \hat{\Delta}_b \tag{51}$$

Here, $\mathbb{A}$ is related to the spatial second moments of the distribution (normalized by a minimum approach distance). Likewise $\mathbb{C}$ are the normalized momentum second moments. These were shown to reduce to the results of Bjorken-Mtingwa in the so-called "Coulomb-Log approximation":

$$\mathbb{K}_{ab} \approx 2L_c \int d\Omega \frac{h_{ab}}{h_3} \tag{52}$$

Combining IBS with Perturbation Theory

The IBS evolution equations are expressed in terms of the second moments of the distribution at a given point in the ring. One can thus express these second moments in terms of invariants and emittances. The expressions in (52) depend only on the matrices $\mathbb{C}_{x,y,z}$. In the uncoupled case, $\mathbb{C}_x$ and $\mathbb{C}_z$ are given by

$$\mathbb{C}_x = \begin{pmatrix} \beta_x & 0 & -\gamma\mathcal{G}_x \\ 0 & 0 & 0 \\ -\gamma\mathcal{G}_x & 0 & \gamma^2\mathcal{H}_x \end{pmatrix} \tag{53}$$

$$\mathbb{C}_z = \begin{pmatrix} \gamma_z\eta_x^2 & \gamma_z\eta_x\eta_y & -\alpha_z\gamma\eta_x \\ \gamma_z\eta_x\eta_y & \gamma_z\eta_y^2 & -\alpha_z\gamma\eta_y \\ -\alpha_z\gamma\eta_x & -\alpha_z\gamma\eta_y & \gamma^2\beta_z \end{pmatrix} \tag{54}$$

We have introduced the definitions

$$\mathcal{G}_{x,y} = \alpha_{x,y}\eta_{x,y} + \beta_{x,y}\eta'_{x,y} \tag{55}$$
$$\tilde{\mathcal{G}}_{x,y} = \gamma_{x,y}\eta_{x,y} + \alpha_{x,y}\eta'_{x,y} \tag{56}$$

Near a sum resonance, one finds

$$\mathbb{C}_1 = \cosh^2(\frac{\theta}{2})\mathbb{C}_x + \sinh^2(\frac{\theta}{2})\mathbb{C}_y + \sinh(\theta)\, \mathbb{C}_c^+,$$
$$\mathbb{C}_2 = \sinh^2(\frac{\theta}{2})\mathbb{C}_x + \cosh^2(\frac{\theta}{2})\mathbb{C}_y + \sinh(\theta)\, \mathbb{C}_c^+,$$

with

$$\mathbb{C}_c^+ = \begin{pmatrix} 0 & \sqrt{\beta_x\beta_y}\cos\phi & 0 \\ \sqrt{\beta_x\beta_y}\cos\phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{57}$$

Using these expressions, for large enough θ, one can show that no IBS equilibrium is possible, even below transition!

CONCLUSIONS

We have described a framework with which to calculate the beam distribution near synchrobetatron coupling resonances. This framework allows the computation of analytical results. We can thus derive many useful results, such as stop-band widths, coupled invariants, coupled emittances, and even coupled emittance evolution in the presence of Intrabeam Scattering.

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