Equilibrium Beam Distribution in an Electron Storage Ring near Linear Synchrobetatron Coupling Resonances

PAC '07

June 26, 2007

Based on Stanford University thesis work with Alex Chao

Also, Nash, Wu, Chao, Phys. Rev. ST Accel. Beams **9**, 032801 (2006).

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Outline

1. Introduction
2. Perturbation Theory Calculation of Invariants
3. Inclusion of damping/diffusion to find emittances
4. Theoretical properties of framework
5. Non-uniform diffusion/damping
6. Conclusions

Introduction

Consider a bunch of electrons in a storage ring:

Linear Symplectic Dynamics

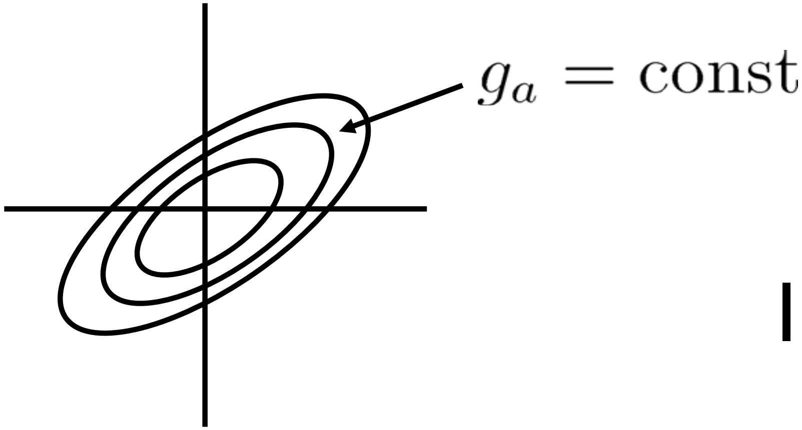
+

Damping/Diffusion Process

Gaussian Beams

$$f(\vec{z}) = \frac{1}{\Gamma} e^{\frac{1}{2} z_i z_j \Sigma^{-1}_{ij}}$$

Gaussian Distributions with Invariants



Invariants: $g_{1,2,3}$

Beam Distribution:

$$f(\vec{z}) = \frac{1}{\pi^3 \langle g_1 \rangle \langle g_2 \rangle \langle g_3 \rangle} \exp\left(-\frac{g_1}{\langle g_1 \rangle} - \frac{g_2}{\langle g_2 \rangle} - \frac{g_3}{\langle g_3 \rangle} \right)$$

$$\langle g_a \rangle = 2\epsilon_a$$

So to find distribution, we need invariants + emittances.

Some previous work

•Sands (1970)

Analytical:

$$\epsilon_x \propto \int \mathcal{H}_x(s)ds$$

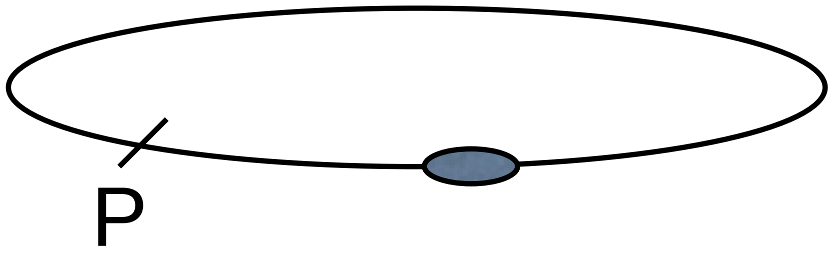
•Chao (1979) (SLIM)

general 6X6 coupled Case
Numerical

•Ohmi *et. al.* (1994) (Beam Envelopes)

Analytical, expressed in terms of fully coupled
Edwards/Teng Lattice parameters

Coupling Effects



$$M_{\text{uncoupled}} = \begin{pmatrix} M_x & 0 \\ 0 & M_{y,z} \end{pmatrix}$$

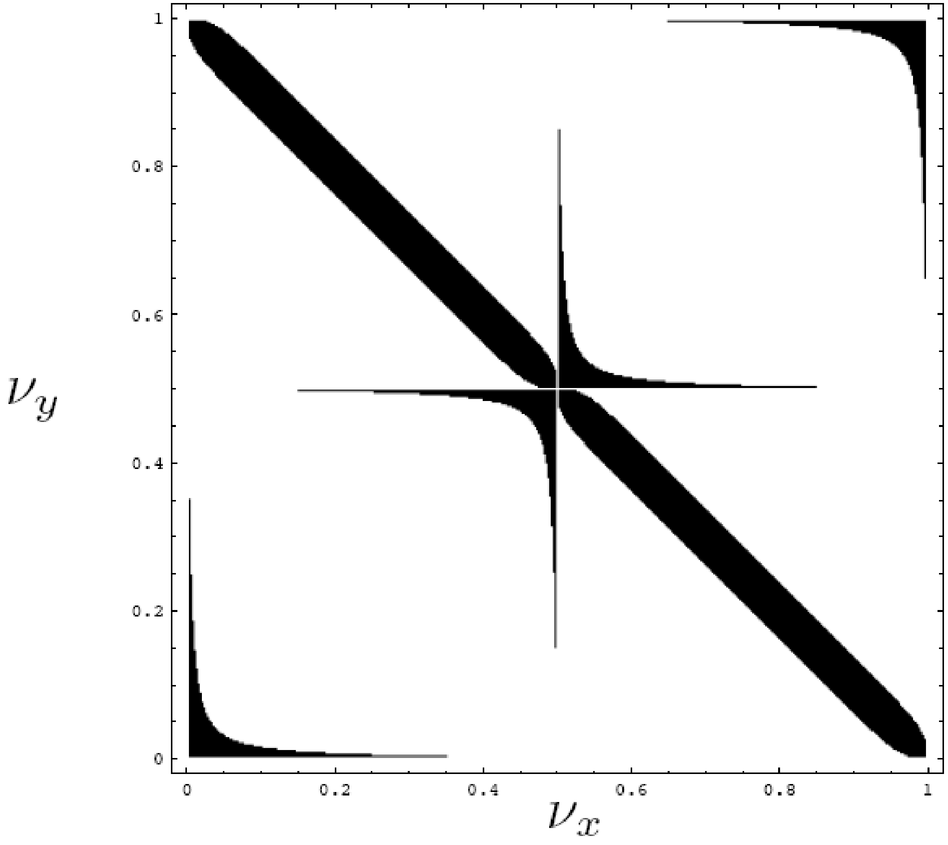
Consider coupling perturbation.

$$T=I+P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \xi_c & 0 \\ 0 & 0 & 1 & 0 \\ \xi_c & 0 & 0 & 1 \end{pmatrix}$$

In x-y space, ξ_c is skew quad strength,
In x-z space, ξ_c is crab cavity strength.

X-Y coupling instability (skew quad)

Instability Plot $\xi_c = .25$



(computed via eigenvalues of exact matrix)

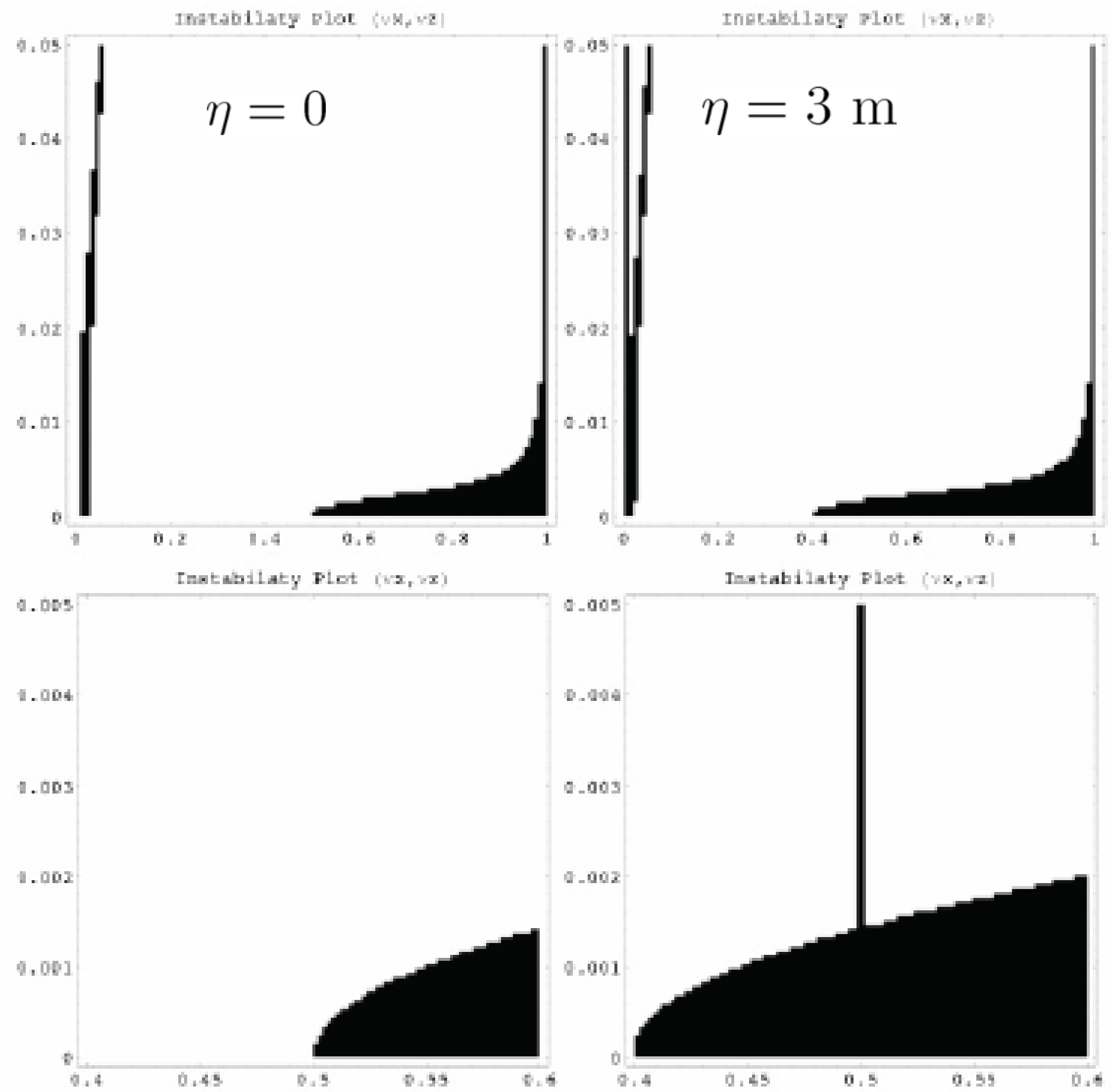
X-Z coupling Instability (crab cavity)

Different, since

$$\beta_z \propto \frac{1}{\nu_s}$$

magnification near half-integer

(KEK operates near 1/2 int.)



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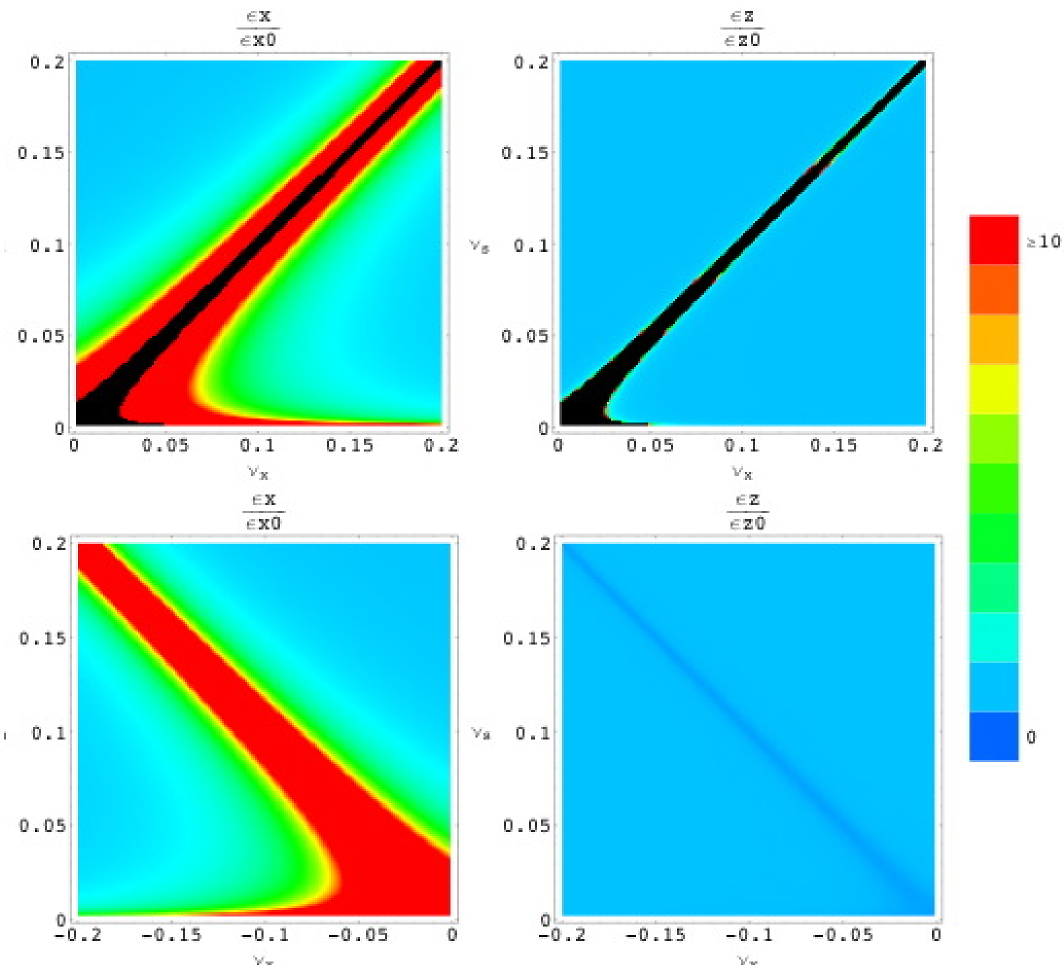
Emittance growth caused by crab cavity near resonances

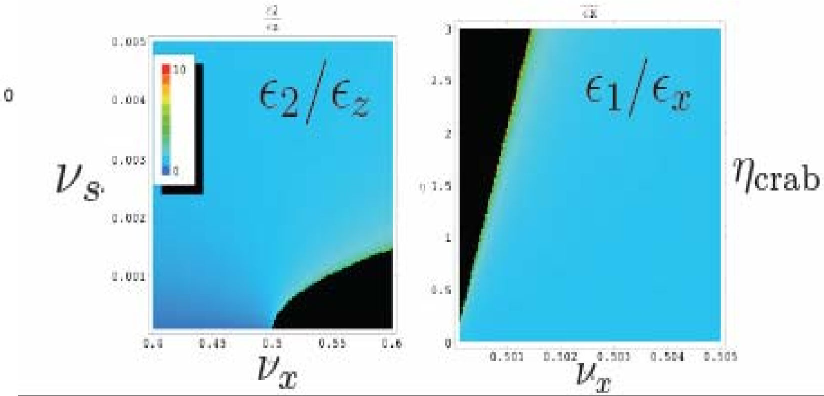
Sum reso.

Diff reso.



X Z

$\nu_s \approx 0$ $\nu_x \approx \frac{1}{2}$



Z X

Params from PEP-II

Computed w/ our Analytical formulas

Linear half integer res.
Doesn’t look bad, sum/dif strong

Non-linear resonances can be a problem– would require extension of this framework

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- Chao/ Ohmi approaches can compute these results (SLIM+, SAD, PTC, etc.) But can we get analytical understanding??

(1)

When are analytical results useful??

	Numerical	Analytical
	Useful for precise calculation of Specific case(s)	Useful for understanding generic properties
	Algorithmic complexity not so important, as long as speed is reasonable.	Results should be simple..

For small coupling, we develop a perturbative approach with simple analytical results.

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(2) # Perturbation Theory Calculation of Invariants

- Develop perturbation theory
- Analogous to Quantum mechanics
- Near resonance means degenerate PT
- Integer/Half Integer resonances due to coupling require special care (2^{nd} order!)

From eigenvectors to Invariants

Theorem:

Given eigenvectors of M

$$G_a = -J(v_a v_a^\dagger + v_a^* v_a^T)J \qquad a = 1, 2, 3$$

$\Rightarrow$ $g_a = \vec{z}^T G_a \vec{z}$ are three invariants of M

Formal Connection to QM

Beam Dynamics	TI Shroedinger Eqn.
$M\vec{v} = \lambda\vec{v}$	$\hat{H}\psi = E\psi$
Eigenvalues give tunes.	Eigenvalues give energies.
Eigenvectors give invariants.	Eigenvectors give stationary states.
M is symplectic.	H is Hermitian.

Uncoupled Storage Ring

$$v_x = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\beta_x} \\ \frac{i-\alpha_x}{\sqrt{\beta_x}} \\ 0 \\ 0 \end{pmatrix} \qquad v_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ \sqrt{\beta_z} \\ \frac{i-\alpha_z}{\sqrt{\beta_z}} \end{pmatrix}$$

Eigenvectors

β_x, α_x are familiar, but what about β_z, α_z ??

$$\beta_z = \frac{a}{\mu_s}, \qquad \gamma_z = \frac{\mu_s}{a}, \qquad \alpha_z = \frac{-\mu_s}{2}(1 - 2\check{\alpha})$$

$a = C\alpha_c$

$\check{\alpha} =$ Partial momentum compaction factor

$$g_a = \gamma_a X_a^2 + 2\alpha_a X_a P_a + \beta_a P_a'^2$$

$a = x, z$ Two invariants

note: Courant Snyder analysis generalized to z-motion

Perturbation

$$M = M_0 + M_1 = (1 + P)M_0$$

M_0 is degenerate unperturbed one-turn map exactly on resonance,

Examples for P:

Dispersion ≠ 0 at an RF cavity

Crab cavity

Find coupling from matrix elements

$$r_{jk} = v^{j0} P v_{k0}$$ ← Matrix elements from the perturbation

Out of these, for each resonance, we compute a

Splitting parameter	Coupling coefficient	Phase parameter
$\Delta\mu$	ξ	ϕ

Coupling angle θ : *tan* θ or *tanh* $\theta = \xi / \Delta\mu$

Example Result for Crab Cavity

- All resonances were analyzed for both crab cavity and RF dispersion.

For a crab cavity for sum/difference res., we find

$$\xi_{\pm} \equiv 2|r_{\mp 12}| = \xi_c \sqrt{\frac{a\beta_x}{\mu_s} \mp 2\eta^2}$$

Stop-band width depends on dispersion, increases for small synchrotron tune.

Rederived result of Hoffstaetter, Chao (2004)

Construct the coupled invariants

Example, near difference resonance:

$$G_1 = \cos^2(\tfrac{\theta}{2})G_x + \sin^2(\tfrac{\theta}{2})G_y + \sin(\theta)G_c^-$$

$$G_2 = \sin^2(\tfrac{\theta}{2})G_x + \cos^2(\tfrac{\theta}{2})G_y - \sin(\theta)G_c^-$$

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Add Damping/Diffusion to Find Emittances

Total change in emittance per turn:

$$\Delta\epsilon_a = -4\chi_a\epsilon_a + \bar{d}_a$$

$$\Longrightarrow \quad \epsilon_{a,eq} = \frac{\bar{d}_a}{4\chi_a}$$

Uncoupled result

Computing damping and diffusion,
we find the emittances:

$$\epsilon_x = \frac{\frac{55}{48\sqrt{3}}\alpha_0\gamma^5 \oint ds \frac{\mathcal{H}_x}{|\rho^3|}}{\frac{2U_0}{E_0}\mathcal{J}_x}$$

$$\epsilon_z = \frac{\frac{55}{48\sqrt{3}}\alpha_0\gamma^5 \frac{a}{\mu_s} \oint ds \frac{1}{|\rho^3|}}{\frac{2U_0}{E_0}\mathcal{J}_z}$$

Reproduces results of Sands using very different (generalizable) approach.

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Sum rules

Invariant sum rule:

$$G_1 \pm G_2 = G_x \pm G_z = \text{invariant}$$

+ = diff. reso. Stability
- = sum reso. Instability

Damping decrement sum rule:

$$\chi_1 + \chi_2 = \chi_x + \chi_z = \text{invariant}$$

(manifestation of Robinson sum rule)

Our framework contains both invariant and Robinson sum rules.

Anti-damping Instability

A surprising result…

Near a sum resonance, one of the damping decrements may become negative. There is an instability for all coupling angles greater than...

$$\coth(\frac{\theta_+}{2}) = \sqrt{\frac{\chi_z}{\chi_x}}$$

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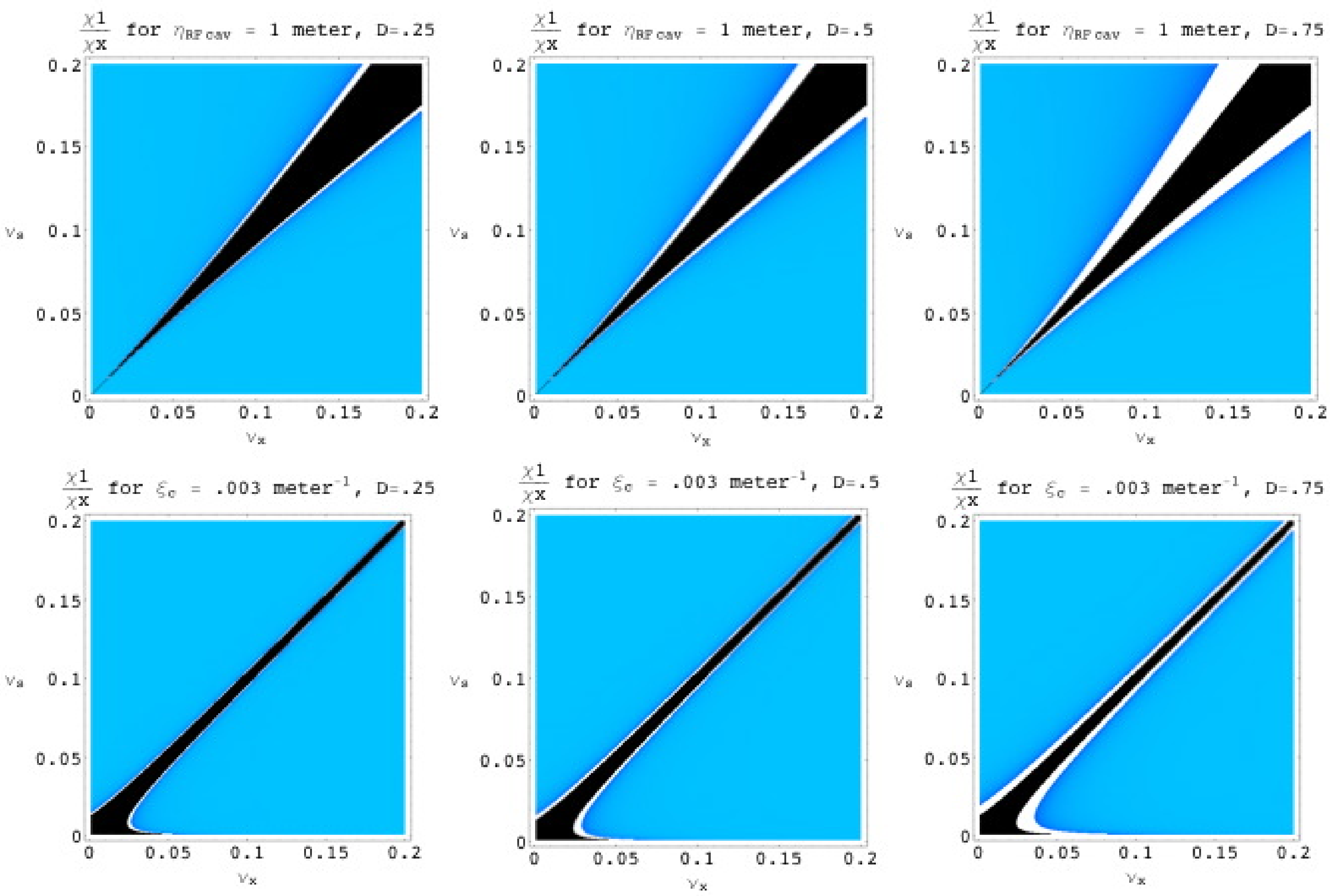
Anti-Damping Instability for Varying Damping Partition Number

RF cav. disp

crab cavity



Emittance Coupling

Near difference resonance:

$$\epsilon_{1,eq} = \frac{\cos^2(\frac{\theta}{2})\bar{d}_x + \sin^2(\frac{\theta}{2})\bar{d}_y}{4(\cos^2(\frac{\theta}{2})\chi_x + \sin^2(\frac{\theta}{2})\chi_y)} = \cos^2\left(\frac{\theta}{2}\right)\epsilon_x + \sin^2\left(\frac{\theta}{2}\right)\epsilon_y$$

$$\epsilon_{2,eq} = \frac{\sin^2(\frac{\theta}{2})\bar{d}_x + \cos^2(\frac{\theta}{2})\bar{d}_y}{4(\sin^2(\frac{\theta}{2})\chi_x + \cos^2(\frac{\theta}{2})\chi_y)} = \sin^2\left(\frac{\theta}{2}\right)\epsilon_x + \cos^2\left(\frac{\theta}{2}\right)\epsilon_y$$

So, emittance coupling,
Not always a rigorous concept!
Does not apply to SB coupling.

Familiar result if
(but only if) $\chi_x = \chi_y$

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Non-uniform diffusion/damping

Non-uniform means the damping and diffusion coefficients depend on phase space position. Important examples are intrabeam scattering and beam-beam.

(A. Chao, AIP Conf. Proc., 127, 201, 1985)

Now we have more general $\epsilon_{1,2,3}(t)$

Intrabeam scattering

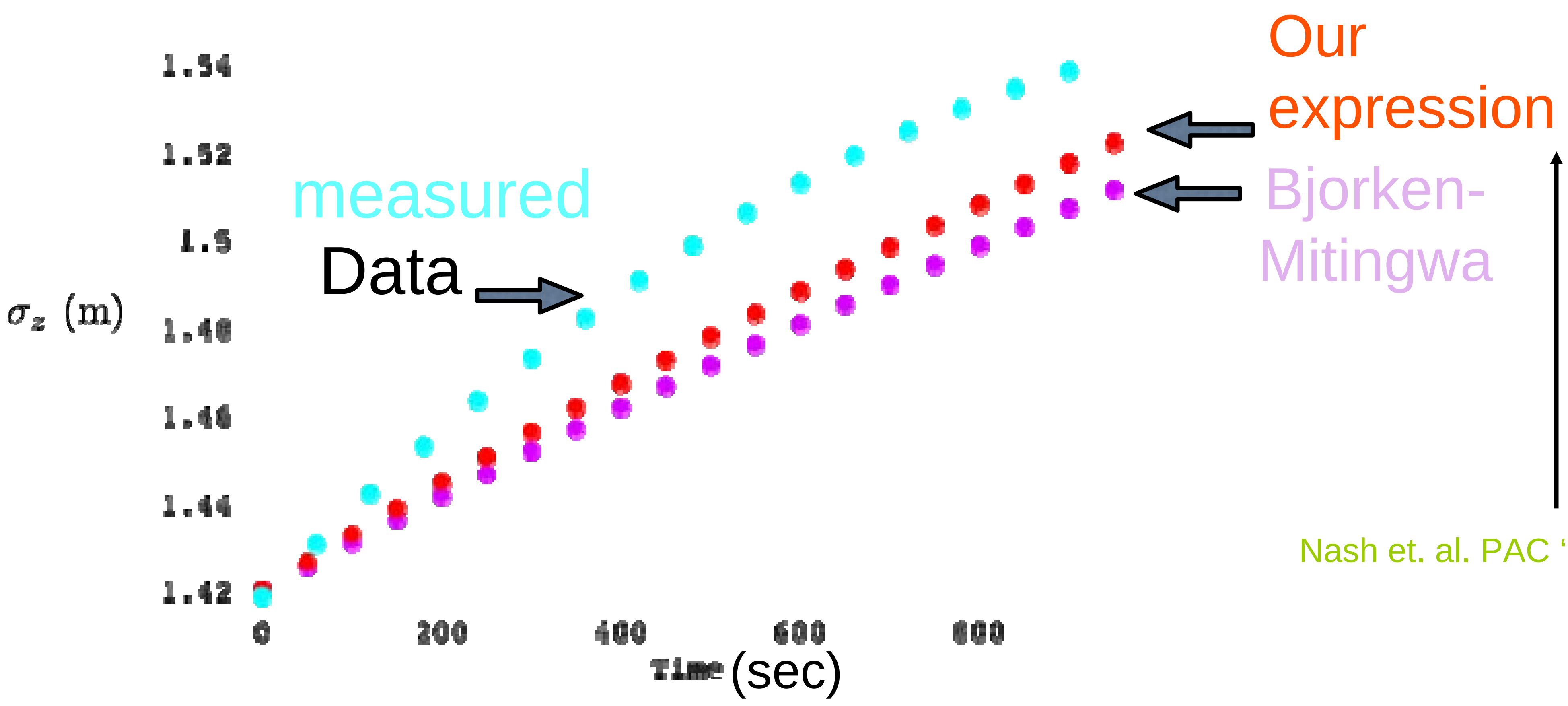
IBS growth depends on distribution!

Same framework applies, still need to find Invariants and emittance evolution.

After finding invariants, solve an equation of the form

$$\frac{d\epsilon_{1,2,3}}{dt} = F(\epsilon_{1,2,3})$$

Example: RHIC, gold ions at injection



Nash et. al. PAC '03

Add Synchrobetatron Coupling?

We can also apply our perturbation theory results to IBS!

To give a rough idea of how this goes:

Beam frame momentum distribution:

$$C_x = \begin{pmatrix} \beta_x & 0 & -\gamma\mathcal{G}_x \\ 0 & 0 & 0 \\ -\gamma\mathcal{G}_x & 0 & \gamma^2\mathcal{H}_x \end{pmatrix} \qquad C_z = \begin{pmatrix} \gamma_z\eta_x^2 & \gamma_z\eta_x\eta_y & -\alpha_z\gamma\eta_x \\ \gamma_z\eta_x\eta_y & \gamma_z\eta_y^2 & -\alpha_z\gamma\eta_y \\ -\alpha_z\gamma\eta_x & -\alpha_z\gamma\eta_y & \gamma^2\beta_z \end{pmatrix}$$

Near difference resonance get replaced with:

$$C_1 = \cos^2\theta C_x + \sin^2\theta C_z + \sin(2\theta)C_c \qquad C_2 = \sin^2\theta C_x + \cos^2\theta C_z - \sin 2\theta C_c$$

Then evolve coupled invariants

(5)

Combine IBS/PT

Now we can explore interaction between IBS and coupling resonances!

Surprising result:

IBS+SBC -> no equilibrium below transition!

Understand vertical emittance due to coupling and vertical dispersion.

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Conclusions and future directions

- Gaussian beam distribution is determined by invariants + emittances
- Perturbation theory to find invariants was developed. Near resonance required degenerate PT.
- The case of a crab cavity was analyzed. Half integer resonance not too strong, dispersion can increase stop-band width and emittance growth.
- General analytical expressions reduce to Sands in uncoupled case.
- Diffusion/damping change emittances. Radiation and IBS have been analyzed in detail. Beam-beam, gas scattering and other diffusive effects could also be included.
- Interaction between resonance and damping/diffusion has a rich phenomenology: anti-damping instability, emittance coupling, beam equilibrium with IBS, etc.

Acknowledgements

- Alex Chao, Juhao Wu, Karl Bane for collaboration
- Wolfram Fischer for RHIC data
- Alexei Blednyck and Johan Bengtsson for discussion and help w/ presentation

Thanks for listening!!

Extra Slides

Damping matrix (for radiation damping)

$$B_\beta = \mathcal{B} B \mathcal{B}^{-1} = \begin{pmatrix} -b_{\delta x}\eta_x & 0 & 0 & 0 & 0 & -b_z\eta_x - b_{\delta x}\eta_x^2 \\ -b_{\delta x}\eta_x' & b_x & 0 & 0 & 0 & (b_x - b_z)\eta_x' - b_{\delta x}\eta_x'\eta_x \\ -b_{\delta x}\eta_y & 0 & 0 & b_y & 0 & (b_y - b_z)\eta_y' - b_{\delta x}\eta_y'\eta_x \\ 0 & -b_x\eta_x & 0 & -b_y\eta_y & 0 & -b_x\eta_x'\eta_x - b_y\eta_y'\eta_y \\ b_{\delta x} & 0 & 0 & 0 & 0 & b_z + b_{\delta x}\eta_x \end{pmatrix}$$

$$b_x(s) = \sum_i \frac{U_{0i}}{cP_0}\delta(s - s_{ci})$$

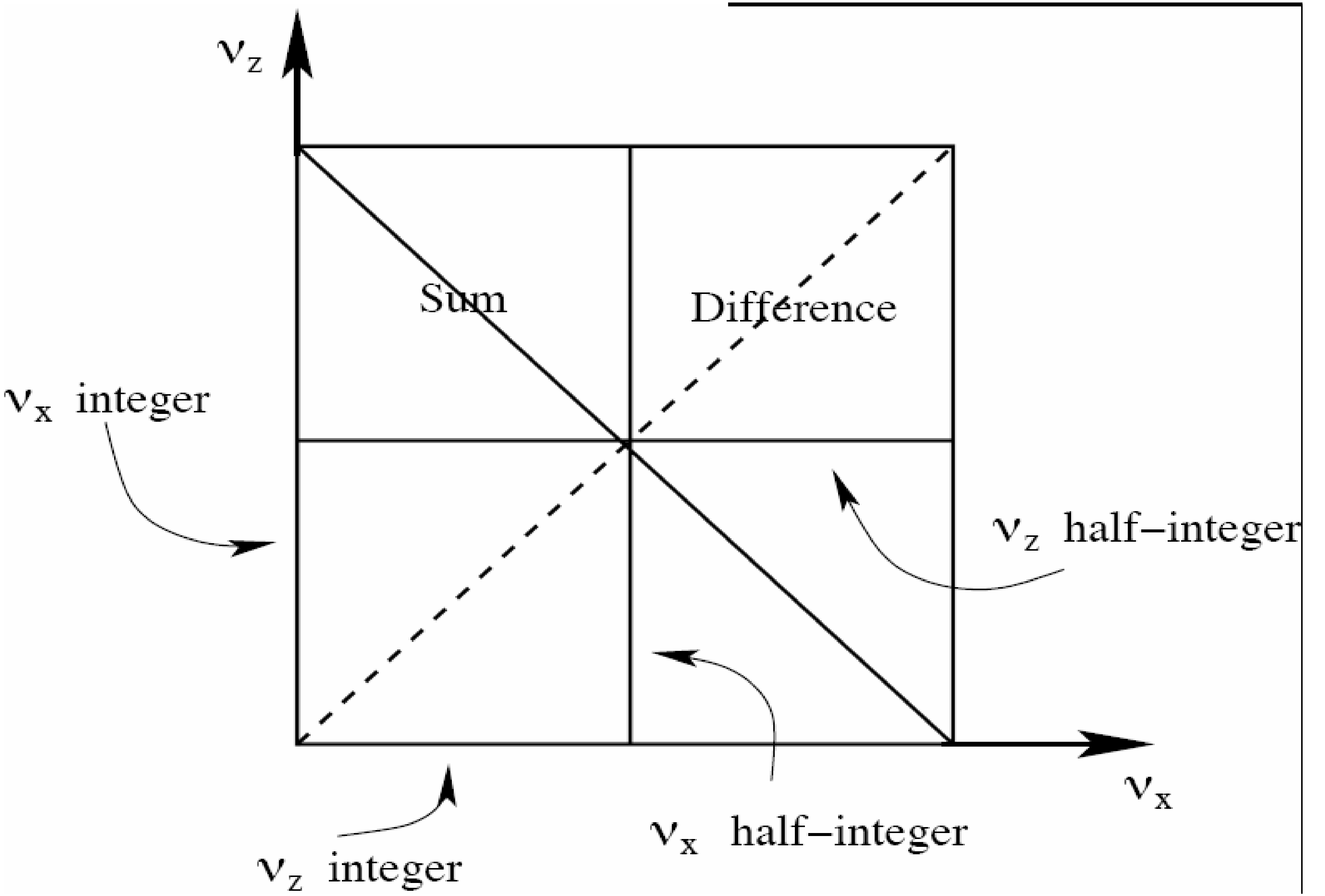
$$b_z = P_\gamma c P_0, \quad b_{\delta x} = \frac{P_\gamma}{2cE_0}\left(\frac{1}{\rho} + \frac{2}{B_y}\frac{\partial B_y}{\partial x}\right),$$

Diffusion matrix (for quantum diffusion)

$$D_\beta = \mathcal{B} D \mathcal{B}^T = d \begin{pmatrix} \eta_x^2 & \eta_x\eta_x' & \eta_x\eta_y & \eta_x\eta_y' & 0 & -\eta_x \\ \eta_x\eta_x' & \eta_x'^2 & \eta_x'\eta_y & \eta_x\eta_y' & 0 & -\eta_x' \\ \eta_x\eta_y & \eta_x'\eta_y & \eta_y^2 & \eta_y\eta_y' & 0 & -\eta_y \\ \eta_x\eta_y' & \eta_x'\eta_y' & \eta_y\eta_y' & \eta_y'^2 & 0 & -\eta_y' \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\eta_x & -\eta_x' & -\eta_y & -\eta_y' & 0 & 1 \end{pmatrix}$$

$$d(s) = \frac{55}{48\sqrt{3}} \alpha_0 \frac{\gamma^5}{|\rho(s)|^3} \left(\frac{\hbar}{mc}\right)^2$$

Linear resonances



PT results for all resonances

Splitting parameter | Coupling coefficient | Phase parameter

res.	condition	$\Delta\mu$ (mod 2π)	ξ	ϕ	μ
sum	$\mu_x+\mu_z=2\pi n$	$\mu_x+\mu_z-i(r_{11}+r_{22})$	$2\lvert r_{12}\rvert$	$\arg(r_{12})$	$i(r_{11}+r_{22})$
diff.	$\mu_x-\mu_z=2\pi n$	$\mu_x-\mu_z-i(r_{11}-r_{22})$	$2 r_{12}$	$\arg(r_{12})$	$-i(r_{11}-r_{22})$
int (x)	$\mu_x=2\pi n$	$2\mu_x-2ir_{11}$	$2\lvert r_{11}\rvert$	$\arg(r_{11})$	0
int (z)	$\mu_z=2\pi n$	$2\mu_z-2ir_{22}$	$2\lvert r_{22}\rvert$	$\arg(r_{22})$	0
$\frac{1}{2}$-int(x)	$\mu_x=\pi(2n+1)$	$2(\mu_x-\pi)-2ir_{11}$	$2\lvert r_{11}\rvert$	$\arg(r_{11})$	0
$\frac{1}{2}$-int(z)	$\mu_z=\pi(2n-1)$	$2(\mu_z-\pi)-2ir_{22}$	$2\lvert r_{22}\rvert$	$\arg(r_{22})$	0
cp. int (x)	$\mu_x=2\pi n$	$2\mu_x-2ir_{11}-(\lvert r_{12}\rvert^2+\lvert r_{21}\rvert^2)\cot(\frac{\mu_z}{2})$	$2\lvert r_{11}-ir_{21}r_{12}\cot(\frac{\mu_z}{2})\rvert$	$\arg(r_{11}+ir_{21}r_{12}\cot(\frac{\mu_z}{2}))$	0
cp. int (z)	$\mu_z=2\pi n$	$2\mu_z-2ir_{22}-(\lvert r_{12}\rvert^2+\lvert r_{21}\rvert^2)\cot(\frac{\mu_x}{2})$	$2\lvert r_{22}-ir_{12}r_{21}\cot(\frac{\mu_x}{2})\rvert$	$\arg(r_{22}-ir_{12}r_{21}\cot(\frac{\mu_x}{2}))$	0
cp. $\frac{1}{2}$-int(x)	$\mu_x=\pi(2n+1)$	$2(\mu_x-\pi)-2ir_{11}+(\lvert r_{12}\rvert^2+\lvert r_{21}\rvert^2)\tan(\frac{\mu_z}{2})$	$2\lvert r_{11}-ir_{21}r_{12}\tan(\frac{\mu_z}{2})\rvert$	$\arg(r_{11}-ir_{12}r_{21}\tan(\frac{\mu_z}{2}))$	0
cp. $\frac{1}{2}$-int(z)	$\mu_z=\pi(2n-1)$	$2(\mu_z-\pi)-2ir_{22}-(\lvert r_{12}\rvert^2+\lvert r_{21}\rvert^2)\tan(\frac{\mu_x}{2})$	$2\lvert r_{22}-ir_{12}r_{21}\tan(\frac{\mu_x}{2})\rvert$	$\arg(r_{22}-ir_{12}r_{21}\tan(\frac{\mu_x}{2}))$	0

PEP-II parameters used in calculations

parameter	value
C	2199.33 m
α_c	1.23×10^{-3}
ν_x	1.19×10^{-1}
ν_z	2.4×10^{-1}
ϵ_x	49.2×10^{-9} m
ϵ_z	9.35×10^{-9} m
$\bar{d}_x$	2.34×10^{-11} m
$\bar{d}_z$	8.98×10^{-9} m
$\beta(s_{cav})$	20 m
$\alpha(s_{cav})$	0
$\beta(s_{crab})$	20 m
$\alpha(s_{crab})$	0
$\eta(s_{cav})$	0-3 m
$\eta'(s_{cav})$	0
$\eta(s_{crab})$	0-3 m
$\eta'(s_{crab})$	0
ξ_c	0-.003 1/m